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Teaching Handout

Geometry of Curves and Surfaces

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Teaching Handout prepared by :
Fethi Latti

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Department of Mathematics

Geometry of Curves and Surfaces

Course handout
for first-year Master students (M1)

Fethi Latti

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Preface

This course document is intended for Master 1 students. Its objective is to enable them to understand and master the geometry of curves and surfaces, a fundamental discipline of modern differential geometry.

This document offers a rigorous and progressive exploration of the foundations of this discipline. It begins with the study of plane and space curves, introducing essential notions such as parametrization, regularity, curvilinear abscissa, curvature, and the Serret-Frenet frame. The local analysis of curves, including the study of singular points and infinite branches, is also detailed.

The second part is devoted to the study of surfaces in space. The emphasis is placed on the fundamental tools: the fundamental forms (first and second), Gaussian curvature, the Gauss map, and the concept of minimal surfaces. Classic and essential examples, such as the sphere, cylinder, catenoid, helicoid, and the surfaces of Scherk and Enneper, are studied to illustrate these theoretical concepts.

Each chapter combines formal definitions, proofs of major results, and a series of solved exercises. The aim is to provide a rigorous and structured framework for mastering the tools of differential geometry, from basic notions of curves and surfaces to advanced methods for the local and global analysis of complex geometric objects.

May students find here the computational methods and interpretative frameworks needed to approach the geometry of submanifolds in Euclidean space and develop a solid geometric intuition.

Contents

Introduction

The geometry of curves and surfaces plays a central role in modern differential geometry, both for its conceptual depth and its wide range of applications. This course document offers a rigorous and progressive exploration of the foundations of this discipline, starting from plane and space curves and advancing to an in-depth study of surfaces and their local and global properties.

The approach focuses on presenting the essential notions in detail: parametrization of curves, regularity, parametrized arcs, local analysis (curvature, Serret–Frenet frame), and classification of singular points and geometric configurations encountered in curve theory. This first part provides a solid basis for understanding more complex objects such as surfaces in space, for which fundamental tools are introduced: the fundamental forms, Gaussian curvature, the Gauss map, and the concept of minimal surfaces.

The material is especially intended for master's students, who will find here the computational methods and interpretative frameworks needed to approach the geometry of submanifolds in Euclidean space. Each chapter combines formal definitions, proofs of major results, and a series of exercises that illustrate theoretical concepts through classic examples (circle, helix, astroid, sphere, catenoid, helicoid, etc.).

The aim of this document is to provide a rigorous and structured framework for mastering the tools of differential geometry—from basic curve and surface notions to advanced methods for the local and global analysis of complex geometric objects.

To further motivate students, it is worthwhile to emphasize the importance of these concepts in contemporary research (modeling, mathematical physics, minimal surfaces theory, topology, etc.) and their fundamental role in the development of modern geometric and analytic mathematics.

Chapter 1

The curves

1.1 Generality

Definition 1.1.1. *We call a parameterized curve in* $\mathbb{R}^n$ *the data of a couple* (I, γ) *where* I *is an interval of* $\mathbb{R}$ *and* $\gamma : I \to \mathbb{R}^n$ *a continuous application* $\gamma \in C(I, \mathbb{R}^n)$

$$\gamma : I \to \mathbb{R}^n t \to (\gamma_1(t), ..., \gamma_n(t))$$

γ *is continuous* $\Leftrightarrow \gamma_i : I \to \mathbb{R}$ *is continuous for* $i = \overline{1, n}$

- $\gamma(I) = \Gamma$ *is the support of the curve* (I, γ)*.*
- *The curve* (I, γ) *is said to be of class* C^p *if* γ *is of class* C^p $(p > 0)$ *i.e., all* γ_i *are of class* C^p*.*

Definition 1.1.2. *Let* $\gamma : I \to \mathbb{R}^n$ *be a parameterized curve of class* C^1*, a point* $M = \gamma(t_0) \in \gamma(t)$ *is said to be regular if and only if* $\|\gamma'(t_0)\| \neq 0$*. The curve* $\gamma : I \to \mathbb{R}^n$ *is said to be regular if all its points are regular* $\|\gamma'(t)\| \neq 0 \ \forall t \in I$*.*
If $\|\gamma'(t_0)\| = 0$*, the point* $M = \gamma(t_0)$ *is said to be a singular or stationary point.*

Example 1.1.1. *Let* $\gamma(t) = (t, 0)$ *and* $\tilde{\gamma}(t) = (t^3, 0)$ *be two curves defined on* $\mathbb{R}$*.* γ *and* $\tilde{\gamma}$ *have the same support but:*

1. γ *is regular for all* $t \in \mathbb{R}$
2. $\tilde{\gamma}$ *is singular for* $t = 0$

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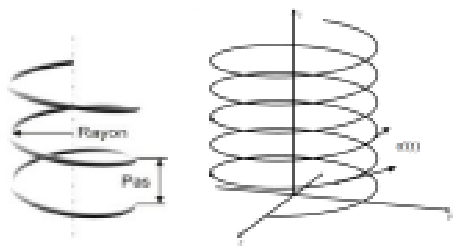
Definition 1.1.3. *Let $\gamma : I \to \mathbb{R}^3$ be a parameterized curve of class C^k. We say that:*

1. *γ is a line if its support $\Gamma = supp(\gamma)$ is contained in a line of $\mathbb{R}^3$. This happens if and only if, for all $t \in I$, all non-zero derivatives $\gamma^{(p)}(t)$ are collinear vectors.*

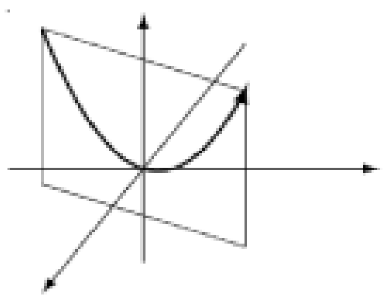
2. *γ is a plane curve if its support $\Gamma = supp(\gamma)$ is contained in a plane of $\mathbb{R}^3$. By assuming without loss of generality, we can suppose that $\gamma : I \to \mathbb{R}^2$.*

3. *γ is a space curve if it is not planar.*

Example 1.1.2. - *The curve $\gamma(t) = (t^5, 3t^5 + 1, 2t^5)$, with $t \in \mathbb{R}$, is a line because its support is the line $\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid y = 3x + 1, z = 2x\}$.*
- *The circular helix is a space curve.*
- *The curve $\gamma(t) = (t, t, t^2)$, with $t \in \mathbb{R}$, is a plane curve because its support $\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid y = x, z = x^2\}$ is a parabola contained in the plane with equation $y = x$.*
- *The graph of any real function $f : I \to \mathbb{R}$ is a parameterized curve $\gamma(t) = (t, f(t))$.*



Space curve (circular helix)



plane curve

1.1.1 Simple points and multiple points

Definition 1.1.4. *Let $\gamma : I \to \mathbb{R}^n$ be a parameterized curve.*

$$\gamma^{-1}(m) = \{t \in I \mid \gamma(t) = m\}$$

If there exists a unique $t \in I$ such that $m = \gamma(t)$, we say that m is a simple point. Otherwise, it is called a multiple point.

Example 1.1.3. *To find the multiple points of the curve $\gamma : (-3, 3) \to \mathbb{R}^2$ given by $t \to (t^3 - 4t, t^2 - 4)$, we need to identify the values of t where the curve intersects itself. This means we are looking for distinct t_1 and t_2 such that:*

$$\gamma(t_1) = \gamma(t_2)$$

So, let's set the coordinates equal to each other and solve for t_1 and t_2:

$$\begin{array}{ll} 1. & t_1^3 - 4t_1 = t_2^3 - 4t_2 \\ 2. & t_1^2 - 4 = t_2^2 - 4 \end{array}$$

From the second equation:

$$t_1^2 = t_2^2$$

This implies that $t_1 = t_2$ or $t_1 = -t_2$.
Now we analyze both cases:
Case 1: $t_1 = t_2$
This would indicate that the point is not a multiple point but rather a simple point.
Case 2: $t_1 = -t_2$
In this case, we substitute $t_1 = -t_2$ into the first equation:

$$(-t_2)^3 - 4(-t_2) = t_2^3 - 4t_2$$

Simplifying the equation:

$$-t_2^3 + 4t_2 = t_2^3 - 4t_2$$

$$-t_2^3 - t_2^3 + 4t_2 + 4t_2 = 0$$

$$-2t_2^3 + 8t_2 = 0$$

Factoring out t_2:

$$t_2(-2t_2^2 + 8) = 0$$

So, $t_2 = 0$ *or* $-2t_2^2 + 8 = 0$*.*
Solving $-2t_2^2 + 8 = 0$*:*

$$-2t_2^2 = -8$$

$$t_2^2 = 4$$

$$t_2 = 2 \text{ or } t_2 = -2$$

Thus, the multiple points occur at:

$$t_2 = 2 \text{ and } t_1 = -2$$

$$t_2 = -2 \text{ and } t_1 = 2$$

Therefore, the multiple points of the curve are:

$$\gamma(2) = (2^3 - 4 \cdot 2, 2^2 - 4) = (8 - 8, 4 - 4) = (0, 0)$$

$$\gamma(-2) = ((-2)^3 - 4 \cdot (-2), (-2)^2 - 4) = (-8 + 8, 4 - 4) = (0, 0)$$

So, the curve intersects itself at the point $(0, 0)$*, which is a multiple point.*

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Example 1.1.4.
Find the multiple points of the curve

$$\gamma : \mathbb{R}^* \to \mathbb{R}^2, \quad t \mapsto \left(t^2 - \frac{2}{t},\ t^2 + \frac{1}{t^2}\right)$$

solution :

We are given the curve:

$$\gamma : \mathbb{R}^* \to \mathbb{R}^2, \quad t \mapsto \left(t^2 - \frac{2}{t},\ t^2 + \frac{1}{t^2}\right)$$

We want to find values $t_1 \neq t_2$ such that $\gamma(t_1) = \gamma(t_2)$. That is, we solve:

$$\begin{cases} t_1^2 - \dfrac{2}{t_1} = t_2^2 - \dfrac{2}{t_2} \\ t_1^2 + \dfrac{1}{t_1^2} = t_2^2 + \dfrac{1}{t_2^2} \end{cases}$$

First equation:

$$t_1^2 - t_2^2 = \frac{2}{t_1} - \frac{2}{t_2} \Rightarrow (t_1 - t_2)(t_1 + t_2) = \frac{2(t_2 - t_1)}{t_1 t_2} \Rightarrow (t_1 + t_2)(t_1 - t_2) = -\frac{2(t_1 - t_2)}{t_1 t_2}$$

Assuming $t_1 \neq t_2$, we cancel $t_1 - t_2$:

$$t_1 + t_2 = -\frac{2}{t_1 t_2} \tag{1}$$

Second equation:

$$t_1^2 - t_2^2 = \frac{1}{t_2^2} - \frac{1}{t_1^2} \Rightarrow (t_1 - t_2)(t_1 + t_2) = \frac{(t_1 - t_2)(t_1 + t_2)}{t_1^2 t_2^2} \Rightarrow 1 = \frac{1}{t_1^2 t_2^2} \Rightarrow t_1^2 t_2^2 = 1 \quad (2)$$

Combining (1) and (2):
From (2): $(t_1 t_2)^2 = 1 \Rightarrow t_1 t_2 = \pm 1$

- If $t_1 t_2 = 1$, then from (1): $t_1 + t_2 = -2$
- If $t_1 t_2 = -1$, then from (1): $t_1 + t_2 = 2$

We solve the quadratic equations:
Case 1: $t_1 t_2 = 1,\ t_1 + t_2 = -2$

$$t^2 + 2t + 1 = 0 \Rightarrow (t+1)^2 = 0 \Rightarrow t = -1 \Rightarrow t_1 = t_2 = -1 \quad \text{(not a multiple point)}$$

Case 2: $t_1 t_2 = -1,\ t_1 + t_2 = 2$

$$t^2 - 2t - 1 = 0 \Rightarrow t = 1 \pm \sqrt{2} \Rightarrow t_1 = 1 + \sqrt{2},\ t_2 = 1 - \sqrt{2} \quad \text{(distinct real solutions)}$$

Conclusion: The curve has a double point at

$$\gamma(1+\sqrt{2}) = \gamma(1-\sqrt{2}) = \left((1+\sqrt{2})^2 - \frac{2}{1+\sqrt{2}},\ (1+\sqrt{2})^2 + \frac{1}{(1+\sqrt{2})^2}\right) = (5, 6)$$

which is a multiple point of the curve.

Definition 1.1.5. *Two curves* (I,γ_1) *and* (J,γ_2) *are said to be equivalent if there exists a homeomorphism:*

$$\phi : I \to J \quad \text{such that} \quad \gamma_1 = \gamma_2 \circ \phi$$

$$\begin{array}{ccc} I & \xrightarrow{\gamma_1} & \mathbb{R}^n \\ & & \\ \phi \searrow & & \uparrow \gamma_2 \\ & & J \end{array}$$

We write $(I,\gamma_1) \sim (J,\gamma_2)$.
The relation $\sim$ *thus defined is an equivalence relation. The equivalence class is called a geometric arc. An element of this equivalence class is called a parameterized representation of the geometric arc.*

Remark 1.1.1. *Two curves* (I,γ_1) *and* (J,γ_2) *are equivalent, meaning that: there exists a homeomorphism* $\phi : I \to J$ *such that* $\gamma_1 = \gamma_2 \circ \phi$. *If* ϕ *is strictly positive, we say that these two curves have the same orientation.*

Definition 1.1.6. *Let $t : I_\theta \to \mathbb{R}$ be a real function defined on an interval I_θ of $\mathbb{R}$. We say that $t : I_\theta \to \mathbb{R}, \theta \mapsto t(\theta)$, is an admissible change if:*

$$\begin{cases} t \text{ is of class } C^1 \\ t'(\theta) = \frac{dt}{d\theta} \neq 0 \end{cases}$$

Proposition 1.1.1. *If $t : \theta \to t(\theta)$ is an admissible change of parameter, then:*

1. *t is a bijective monotone function from I_θ to $I_t = t(I_\theta)$*
2. *The inverse function $\theta : t \to \theta(t)$ is also an admissible change of parameter.*

proof :
1. If $t : \theta \to t(\theta)$ is an admissible change of class C^1, then

$$\frac{dt}{d\theta} = t'(\theta) \neq 0$$

is a continuous function. Since $t'(\theta) = \frac{dt}{d\theta} \neq 0$, $t : \theta \to t(\theta)$ is strictly monotone, hence a bijection.

2. The inverse function $\theta : t \to \theta(t)$ is also of class C^1, and we have

$$\theta'(t) = \frac{d\theta}{dt} \neq 0$$

. Therefore, $\theta : t \to \theta(t)$ is also an admissible change of parameter.

Remark 1.1.2.
1. *If $t : \theta \to t(\theta)$ is of class C^k, then $\theta : t \to \theta(t)$ is of class C^k (for $k \geq 1$)*
2. *If $\frac{dt}{d\theta} > 0$, we say that $\theta : t \to \theta(t)$ preserves the orientation*
3. *If $\frac{dt}{d\theta} < 0$, we say that $\theta : t \to \theta(t)$ reverses the orientation*

We introduce the following relation on regular parameterized curves of class C^k (for $k \geq 1$):

$$(I_t, \alpha) \sim (I_\theta, \beta)$$

which means there exists an admissible change of parameter on I_θ such that:

$$\begin{cases} t(I_\theta) = I_t \\ t : \theta \to t(\theta) \text{ is of class } C^k \\ \alpha(t(\theta)) = \beta(\theta) \end{cases}$$

$$\begin{tikzcd} I_t \arrow[r, "\alpha"] & \mathbb{R}^n \\ & I_\theta \arrow[ul, "t"] \arrow[u, "\beta"'] \end{tikzcd}$$

The relation thus defined is an equivalence relation.

Definition 1.1.7. *A regular arc of class C^k (for $k \geq 1$) is an equivalence class of regular parametric representation of class C^k. We also say a regular curve instead of a regular arc.*

Example 1.1.5. *The parameterized curves:*

$$\gamma : (-1, 1) \to \mathbb{R}^2, \quad t \mapsto (t, \sqrt{1 - t^2})$$

$$\alpha : (0, \pi) \to \mathbb{R}^2, \quad \theta \mapsto (\cos\theta, \sin\theta)$$

are equivalent.
Solution :
Indeed, the application

$$\phi : (0, \pi) \to (-1, 1), \quad \theta \mapsto \cos\theta$$

is a C^∞-diffeomorphism because the cosine function is of class C^∞ and strictly decreasing on $(0, \pi)$, and its inverse, arccos, is of class C^∞.
Moreover, $\alpha = \gamma \circ \phi$, thus ϕ is a reparameterization. That is, γ and α are equivalent.
For all $\theta \in (0, \pi)$,

$$\phi'(\theta) = -\sin\theta < 0$$

. Therefore, γ and ϕ define opposite orientations.

1.1.2 Curvilinear Abscissa

How to measure the length of a curve?

A natural way to proceed is to approximate this length by the length of a polygonal line whose vertices are on the curve. The length of the polygonal line is clearly less than that of the curve, but one can easily imagine that if more vertices are added by decreasing the distance between two consecutive vertices, the length of the polygonal line will tend to the length of the curve. The definition of the length of a curve is based on this idea.

Definition 1.1.8. *The length of a parameterized curve* $\gamma : [a, b] \to \mathbb{R}^d$ *(where* $d = 2$ *or* 3*) is given by:*

$$l(\gamma) = \sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{k=0}^{n-1} \|\overrightarrow{\gamma(t_i)\gamma(t_{i+1})}\|$$

where the supremum is taken over all subdivisions $a = t_0 < t_1 < \cdots < t_n = b$ *of the interval* $[a, b]$*, with* n *being any number.*
Moreover, if $l(\gamma)$ *is finite, the curve* γ *is said to be rectifiable.*

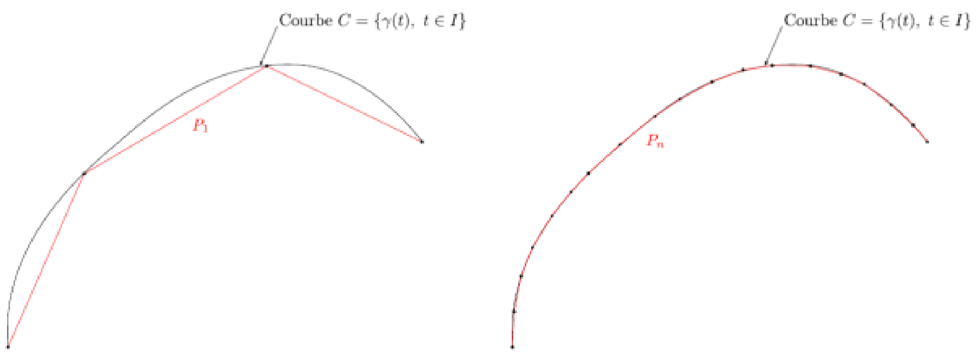


Figure 1.3

Remark 1.1.3. *The length does not depend on the parameterization. The previous definition is geometric and corresponds to our intuition of length. That said, it is not necessarily practical for performing calculations. The following theorem expresses the length of a parameterized curve by an integral formula.*

Theorem 1.1.1. *Let $\gamma : [a, b] \to \mathbb{R}^3$ be a parameterized curve of class C^1. Then γ is rectifiable and we have:*

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt$$

proof:

Consider a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. For every $i \in \{0, 1, \ldots, n-1\}$, we have:

$$\|\gamma(t_{i+1}) - \gamma(t_i)\| = \left\| \int_{t_i}^{t_{i+1}} \gamma'(t)\, dt \right\| \leq \int_{t_i}^{t_{i+1}} \|\gamma'(t)\|\, dt$$

Summing over i, this implies:

$$\sum_{k=0}^{n-1} \|\overrightarrow{\gamma(t_i)\gamma(t_{i+1})}\| \leq \int_a^b \|\gamma'(t)\|\, dt$$

Taking the supremum over all subdivisions, we get that γ is rectifiable and satisfies:

$$l(\gamma) \leq \int_a^b \|\gamma'(t)\|\, dt$$

Now, introducing the function $\varphi : [a, b] \to \mathbb{R}$ which gives the length of the curve between parameters a and t:

$$\forall t \in [a, b], \quad \varphi(t) = l(\gamma|_{[a,t]})$$

Let $t \in [a, b]$ and h such that $t + h \in [a, b]$. The length of the curve between the parameters t and $t+h$ is longer than the segment connecting $\gamma(t)$ to $\gamma(t+h)$, so we have:

$$\frac{1}{h}\|\overrightarrow{\gamma(t)\gamma(t+h)}\| \leq \frac{\varphi(t+h) - \varphi(t)}{h}$$

This leads to the following inequality:

$$\frac{1}{h}\|\overrightarrow{\gamma(t)\gamma(t+h)}\| \leq \frac{\varphi(t+h) - \varphi(t)}{h} \leq \frac{1}{h} \int_t^{t+h} \|\gamma'(u)\|\, du$$

Both the left and right members have the same limit $\|\gamma'(t)\|$, which implies that φ is differentiable at t and $\varphi'(t) = \|\gamma'(t)\|$.

□

Example 1.1.6. *The length of the helix parameterized by $\gamma(t) = (R\cos(t), R\sin(t), at)$ with $\gamma : [0, 2\pi] \to \mathbb{R}^3$ is given by:*

$$l(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\, dt = \int_0^{2\pi} \sqrt{R^2 + a^2}\, dt = 2\pi\sqrt{R^2 + a^2}$$

Notice that if $a = 0$, we find the length of a circle of radius R to be $2\pi R$.

Exercise 1.1.1. ***Length of the Astroid***

We consider the astroid defined by:

$$\gamma(t) = (\cos^3 t, \sin^3 t), \quad t \in [0, 2\pi]$$

We use the standard formula for the length of a plane curve:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

In our case:

$$x(t) = \cos^3 t, \quad y(t) = \sin^3 t$$

We compute the derivatives:

$$x'(t) = \frac{d}{dt}(\cos^3 t) = 3\cos^2 t(-\sin t) = -3\cos^2 t \sin t$$

$$y'(t) = \frac{d}{dt}(\sin^3 t) = 3\sin^2 t \cos t$$

Then:

$$(x'(t))^2 + (y'(t))^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t = 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)$$

Since $\cos^2 t + \sin^2 t = 1$*, we have:*

$$(x'(t))^2 + (y'(t))^2 = 9\cos^2 t \sin^2 t \Rightarrow \sqrt{(x'(t))^2 + (y'(t))^2} = 3|\cos t \sin t|$$

Thus, the total length is:

$$L = \int_0^{2\pi} 3|\cos t \sin t|\, dt$$

Using the identity:

$$|\cos t \sin t| = \left|\frac{1}{2}\sin(2t)\right| = \frac{1}{2}|\sin(2t)|$$

So:

$$L = 3\int_0^{2\pi} |\cos t \sin t|\, dt = \frac{3}{2}\int_0^{2\pi} |\sin(2t)|\, dt$$

Since $|\sin(2t)|$ *has period* π*, we get:*

$$\int_0^{2\pi} |\sin(2t)|\, dt = 2\int_0^{\pi} |\sin(2t)|\, dt = 4\int_0^{\frac{\pi}{2}} \sin(2t)\, dt$$

Because $\sin(2t) \geq 0$ *on* $[0, \frac{\pi}{2}]$*, we find:*

$$\int_0^{\frac{\pi}{2}} \sin(2t)\, dt = -\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}[\cos(\pi) - \cos(0)] = -\frac{1}{2}(-1-1) = 1$$

Therefore:

$$L = \frac{3}{2} \cdot 4 \cdot 1 = 6$$

Conclusion: *The length of the astroid is: 6*

1.1.3 Parameterization by Curvilinear Abscissa

Definition 1.1.9. *A parameterization $\gamma : I \to \mathbb{R}^d$ of a geometric curve is said to be normal (or by curvilinear abscissa) if for all $[t_1, t_2] \in I$, the length of the geometric curve between the points $\gamma(t_1)$ and $\gamma(t_2)$ is exactly $t_2 - t_1$:*

$$l(\gamma|_{[t_1,t_2]}) = t_2 - t_1$$

In practice, we do not necessarily have a normal parameterization. If we want to have one, we need to reparameterize. For this, we need the notion of curvilinear abscissa.

Definition 1.1.10. *Let $\gamma : I \to \mathbb{R}^d$ (where $d = 2$ or 3) be a parameterized curve of class C^1 and $t_0 \in I$. The curvilinear abscissa from the parameter point t_0 is the function $s_{t_0} : I \to \mathbb{R}$ given by:*

$$\forall t \in I, \quad s_{t_0}(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du$$

Geometrically, $s_{t_0}(t)$ is the length of the geometric curve $C = \gamma(I)$ between the points $\gamma(t_0)$ and $\gamma(t)$. The following result tells us that any regular parameterized curve of class C^1 can be reparameterized by curvilinear abscissa.

Theorem 1.1.2. *Let $\gamma : I \to \mathbb{R}^d$ be a regular parameterized curve of class C^1 and $t_0 \in I$. Then the curvilinear abscissa $s_{t_0}^{-1} : J \to I$ is an admissible variable change, and*

$$\tilde{\gamma} = \gamma \circ s_{t_0}^{-1} : J \to \mathbb{R}^d$$

is a normal parameterization that has the same geometric support as γ.

Proof. Since γ is C^1 and regular, the map $t \mapsto \|\gamma'(t)\|$ is continuous and strictly positive on I. Therefore, the function

$$s_{t_0}(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du$$

is strictly increasing and of class C^1, hence a bijection onto its image $J = s_{t_0}(I)$. Moreover, by the inverse function theorem, the inverse $s_{t_0}^{-1} : J \to I$ is also of class C^1.

Define the reparameterization $\tilde{\gamma} : J \to \mathbb{R}^d$ by

$$\tilde{\gamma}(s) = \gamma\big(s_{t_0}^{-1}(s)\big).$$

We compute its derivative using the chain rule:

$$\tilde{\gamma}'(s) = \gamma'\big(s_{t_0}^{-1}(s)\big) \cdot \frac{d}{ds} s_{t_0}^{-1}(s).$$

But

$$\frac{d}{ds} s_{t_0}^{-1}(s) = \frac{1}{s_{t_0}'(t)}\Big|_{t=s_{t_0}^{-1}(s)} = \frac{1}{\|\gamma'(t)\|}\Big|_{t=s_{t_0}^{-1}(s)}.$$

Hence,

$$\tilde{\gamma}'(s) = \frac{\gamma'\big(s_{t_0}^{-1}(s)\big)}{\|\gamma'\big(s_{t_0}^{-1}(s)\big)\|}.$$

Therefore, $\tilde{\gamma}$ is parameterized by arc length since for all $s \in J$,

$$\|\tilde{\gamma}'(s)\| = 1.$$

This shows $\tilde{\gamma}$ is a normal parameterization of the same geometric curve as γ.

□

Corollary 1.1.1. *Let $\gamma : [a, b] \to \mathbb{R}^d$ be a regular curve of class C^1. Let $l = l(\gamma)$. Then the curve*

$$\tilde{\gamma} = \gamma \circ s_a^{-1} : [0, l] \to \mathbb{R}^d$$

is a normal reparameterization of γ.

Proposition 1.1.2. *Let $\gamma : I \to \mathbb{R}^d$ be a parameterized curve of class C^1. Then we have:*

$$\textit{the parameterization } \gamma \textit{ is normal} \iff \forall s \in I, \|\gamma'(s)\| = 1$$

Proof. If the parameterization is normal, then the curvilinear abscissa from the point t_0 of γ satisfies for all $t \in I$:

$$s_{t_0}(t) = \int_{t_0}^{t} \|\gamma'(u)\|\, du = t - t_0$$

By differentiating, this gives for all $t \in I$:

$$s_{t_0}'(t) = \|\gamma'(t)\| = 1$$

Conversely, if for all $s \in I$ we have $\|\gamma'(s)\| = 1$, then for all $[t_1, t_2] \subset I$:

$$l(\gamma|_{[t_1,t_2]}) = \int_{t_2}^{t_1} \|\gamma'(s)\|\, ds = t - t_0$$

□ □

Example 1.1.7. *Consider the curve*

$$\gamma : (0,1) \to \mathbb{R}^2$$

$$t \mapsto (t, \sqrt{1-t^2})$$

The curvilinear abscissa $s_0 : (0,1) \to \mathbb{R}$ *is given by:*

$$s_0(t) = \int_0^t \|\gamma'(u)\|\, du = \int_0^t \frac{1}{\sqrt{1-u^2}}\, du = \arcsin(t)$$

The function $\arcsin : (0,1) \to \left(0, \frac{\pi}{2}\right)$ *is a bijection.*
The reparameterization $\tilde{\gamma} = \gamma \circ s_0^{-1} : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2$ *is given by:*

$$\tilde{\gamma}(s) = \gamma(\sin(s)) = (\sin(s), \sqrt{1-\sin^2(s)}) = (\sin(s), \cos(s))$$

•

Example 1.1.8.

Let the curve be defined by:

$$\gamma(t) = (R\cos t, R\sin t), \quad \text{with } R > 0$$

We compute the arc length from 0 *to* t*:*

$$s_0 = \int_0^t \|\gamma'(u)\|\, du = \int_0^t R\, du = Rt \neq t$$

Thus, we have:

$$s(t) = Rt \quad \Rightarrow \quad t = \frac{s}{R}$$

Now we define the reparametrized curve:

$$\tilde{\gamma}(s) = \gamma \circ s_0^{-1}(s) = \gamma\left(\frac{s}{R}\right) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\right)$$

Compute the norm of the derivative:

$$\|\tilde{\gamma}'(s)\| = \sqrt{\left(-\sin\left(\frac{s}{R}\right)\right)^2 + \left(\cos\left(\frac{s}{R}\right)\right)^2} = \sqrt{1} = 1$$

Conclusion: *The reparametrized curve* $\tilde{\gamma}(s)$ *is unit-speed (i.e., parametrized by arc length).*

1.2 Plane Curves

The affine plane $\mathbb{R}^2$ is related to an orthonormal coordinate system $(O, \vec{i}, \vec{j})$. A point M with coordinates (x, y) is the point $M = O + x\vec{i} + y\vec{j}$.

Definition 1.2.1. *Let*

$$\begin{array}{ccc} \gamma & : & I \to \mathbb{R}^2 \\ t & \mapsto & (X(t), Y(t)) \end{array}$$

be a function defined on an interval $I \subset \mathbb{R}$. The set $C = \gamma(I)$ of points $M(t)$ with coordinates $(X(t), Y(t))$, as t describes I, is called the curve by γ.
We denote it as

$$C : \begin{cases} x = X(t) \\ y = Y(t) \end{cases}$$

Example 1.2.1. *Let $f : I \to \mathbb{R}$ be a function with real values, then its representative curve can be seen as the plane curve by*

$$\begin{cases} x = t \\ y = f(t) \end{cases} \qquad (t \in \mathbb{R})$$

The circle centered at $A(x_A, y_A)$ with radius r is parameterized by:

$$\begin{cases} X(t) = x_A + r\cos(t) \\ Y(t) = y_A + r\sin(t) \end{cases} \qquad t \in [0, 2\pi]$$

•

1.2.1 Symmetries, Class of the Curve, Variation

Proposition 1.2.1. *Assume that I is an interval symmetric with respect to the origin. Let X and Y be functions. The invariance of the curve C with respect to symmetry is as follows:*

a remetre tableau AAAAAAAAAAAAAAAAAAAAAAAAAAAAAA AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA

X		Symmetry with respect to (Ox)
Y	Symmetry with respect to (Oy)	Symmetry with respect to (O)

AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA AAAAAAAAAAAAa Fin AAAAAAAAAAAAAAAAAA

Example 1.2.2. *Let* $\gamma(t) = (2\cos(t), \sin(3t))$, $t \in \mathbb{R}$. γ *is* 2π *periodic, so we obtain the complete curve when* t *ranges over* $[-\pi, \pi]$.

$$\begin{cases} x(t) = 2\cos(t) \\ y(t) = \sin(3t) \end{cases} \qquad \textit{for } t \in [-\pi, \pi]$$

We have:

$$\gamma(-t) = (x(-t), y(-t)) = (x(t), -y(t))$$

We will study it on the interval $[0, \pi]$ *and make a symmetry with respect to the* x*-axis. We notice that:*

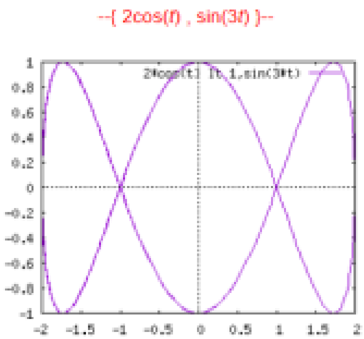
$$\begin{cases} x(\pi - t) = -2\cos(t) = -x(t) \\ y(\pi - t) = \sin(3t) = y(t) \end{cases}$$

We will study it on the interval $[0, \frac{\pi}{2}]$ *and make a symmetry with respect to the* y*-axis.*

Variation Table :

$$\gamma'(t) = \begin{pmatrix} -2\sin(t) \\ 3\cos(3t) \end{pmatrix}$$

t	0		$\frac{\pi}{6}$		$\frac{\pi}{2}$
$x'(t)$	0	$-$	$\vdots$	$-$	
$x(t)$	2	$\searrow$	$\sqrt{3}$ $\sqrt{3}$	$\searrow$	0
$y'(t)$	$+$	$+$	0	$-$	
$y(t)$	0	$\nearrow$	1	$\searrow$	-1



$\gamma(t) = (2\cos(t), \sin(3t))$

•

Example 1.2.3. *Determine the simplest possible domain of study for the curve given by:*

$$\begin{cases} x(t) = t - \dfrac{3}{2}\sin t, \\ y(t) = 1 - \dfrac{3}{2}\cos t \end{cases}$$

Solution

For $t \in \mathbb{R}$, we have:

$$\begin{aligned} \gamma(t+2\pi) &= \left(t + 2\pi - \frac{3}{2}\sin(t+2\pi),\ 1 - \frac{3}{2}\cos(t+2\pi)\right) \\ &= \left(t - \frac{3}{2}\sin t,\ 1 - \frac{3}{2}\cos t\right) + (2\pi, 0) \\ &= \gamma(t) + \vec{u} \end{aligned}$$

where $\vec{u} = (2\pi, 0)$.

Therefore, we study the arc and plot its support over an interval of length 2π, for instance $[-\pi, \pi]$, and obtain the full curve by translating this arc by vectors

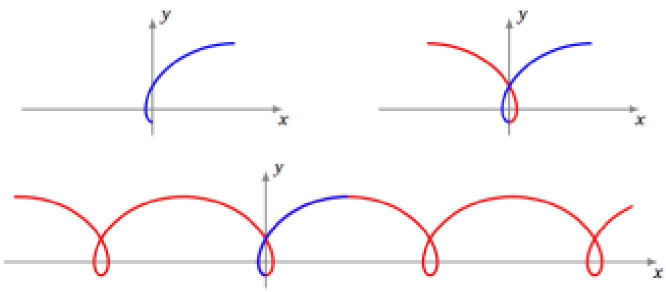
$$k \cdot \vec{u} = (2k\pi, 0), \quad k \in \mathbb{Z}.$$

For $t \in [-\pi, \pi]$, we compute:

$$\gamma(-t) = \left(-t + \frac{3}{2}\sin t,\ 1 - \frac{3}{2}\cos t\right) = s_{Oy}(\gamma(t)).$$

That is, $\gamma(-t)$ is the reflection of $\gamma(t)$ across the y-axis.

So we first study and plot the arc over $[0, \pi]$ (first figure), then perform the reflection across the y-axis (second figure), and finally construct the full curve by translating with vectors $k\vec{u}$, $k \in \mathbb{Z}$ (third figure).



$$\gamma(t) = (t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t)$$

●

1.2.2 Tangents and Points of Inflexion

Proposition 1.2.2. *Let $\gamma : I \to \mathbb{R}^2$ be a regular parameterized curve. Then γ has a tangent directed by the vector $\gamma'(t_0)$ at every $t_0 \in I$.*

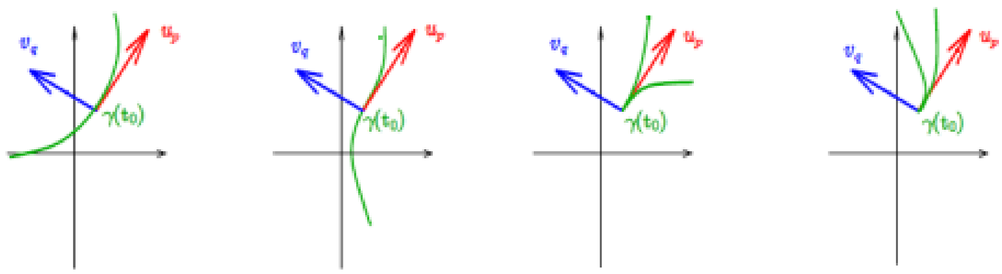
proof :
Since γ is differentiable for all $t_0 \in \mathbb{R}$, then

$$\gamma(t) = \gamma(t_0) + (t - t_0)\gamma'(t_0) + (t - t_0)\epsilon(t) \quad \text{and} \quad \lim_{t\to t_0} \epsilon(t) = (0, 0)$$

Proposition 1.2.3. *Let $\gamma : I \to \mathbb{R}^2$ be a stationary parameterized curve at t_0. Suppose there exist two integers $q > p \geq 2$ and two non-collinear vectors u_p and v_q such that:*

$$\gamma(t) = \gamma(t_0) + (t - t_0)^p u_p + (t - t_0)^q v_q + (t - t_0)\epsilon(t) \quad \text{and} \quad \lim_{t\to t_0} \epsilon(t) = (0, 0)$$

1. *If p is odd and q is even, the curve γ has an ordinary point at t_0*
2. *If p is odd and q is odd, the curve γ has a point of inflexion at t_0*
3. *If p is even and q is odd, the curve γ has a cusp of the first kind at t_0*
4. *If p is even and q is even, the curve γ has a cusp of the second kind at t_0*



The appearance of a stationary point

Example 1.2.4.

1. ***Study of the singular point at $t = 0$ of the parametric curve:***

$$\gamma(t) = \begin{cases} x(t) = t^2 \\ y(t) = t^2 + t^3 \end{cases}$$

The Taylor series expansion

$$\gamma(t) = t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t^3\varphi(t)$$

shows that it is a first type cusp point. The curve possesses a half-tangent with direction vector:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. ***Study of the singular point at*** $t = 0$ ***of the parametric curve:***

$$\gamma(t) = \begin{cases} x(t) = -t^3 + t^4 \\ y(t) = t^3 \end{cases}$$

The Taylor series expansion

$$\gamma(t) = t^3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + t^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t^4 \varphi(t)$$

shows that it is an ordinary point, with a tangent of direction vector:

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

3. ***Study of the singular point at*** $t = 0$ ***of the parametric curve:***

$$\gamma(t) = \begin{cases} x(t) = 3(\sin(t) - t) \\ y(t) = t^3 + t^5 \end{cases}$$

The Taylor series expansion

$$\gamma(t) = t^3 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + t^5 \begin{pmatrix} \frac{1}{40} \\ 1 \end{pmatrix} + t^5 \varphi(t)$$

shows that it is an inflection point, with a tangent of direction vector:

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

4. ***Study of the singular point at*** $t = 0$ ***of the parametric curve:***

$$\gamma(t) = \begin{cases} x(t) = 3(\cos(t) - 1) \\ y(t) = t^2 + t^4 + t^5 \end{cases}$$

The Taylor series expansion

$$\gamma(t) = t^2 \begin{pmatrix} 3 \\ 21 \end{pmatrix} + t^4 \begin{pmatrix} -\frac{1}{8} \\ 1 \end{pmatrix} + t^4 \varphi(t)$$

shows that it is a second type cusp point, with a half-tangent of direction vector:

$$\begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$$

•

Exercise 1.2.1. *Study the curve* γ *defined by:*

$$\gamma(t) = \begin{cases} x(t) = \cos^3 t, \\ y(t) = \sin t \end{cases}$$

Solution: Consider the curve defined by:

$$\gamma(t) = \begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \end{cases}$$

1. Since $\gamma(t + 2\pi) = \gamma(t)$, the curve is 2π-periodic. We analyze it on $[-\pi, \pi]$.

   Symmetries:

   - About the x-axis:

     $$\gamma(-t) = \begin{cases} x(-t) = \cos^3 t = x(t) \\ y(-t) = -\sin^3 t = -y(t) \end{cases}$$

     We analyze it on $[0, \pi]$.
   - About the y-axis:

     $$\gamma(\pi - t) = \begin{cases} x(\pi - t) = \cos^3(\pi - t) = -x(t) \\ y(\pi - t) = -\sin^3(\pi - t) = y(t) \end{cases}$$

     We analyze it on $[0, \frac{\pi}{2}]$.
   - About the first bisector $x = y$:

     $$\gamma\left(\frac{\pi}{2} - t\right) = \begin{cases} x\left(\frac{\pi}{2} - t\right) = \cos^3\left(\frac{\pi}{2} - t\right) = y(t) \\ y\left(\frac{\pi}{2} - t\right) = -\sin^3\left(\frac{\pi}{2} - t\right) = x(t) \end{cases}$$

     We analyze it on $[0, \frac{\pi}{4}]$.

2. Derivative and Stationary Point Analysis: For $t \in [0, \frac{\pi}{4}]$:

   $$\gamma'(t) = \begin{cases} x'(t) = -3(\sin t)(\cos^2 t) \\ y'(t) = 3(\cos t)(\sin^2 t) \end{cases}$$

t	0		$\frac{\pi}{4}$
$x'(t)$	0	$-$	
$x(t)$	1	$\searrow$	$\frac{\sqrt{2}}{4}$
$y(t)$	0	$\nearrow$	$\frac{\sqrt{2}}{4}$
$y'(t)$	0	$+$	

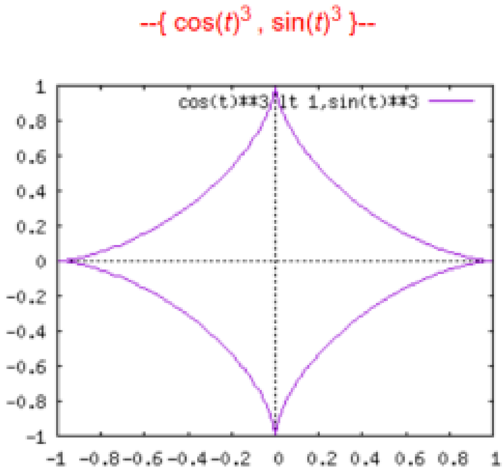
   At $t = 0$,

   $$\gamma'(0) = (0, 0) \quad \text{(singular point)}$$

   Second and third derivatives:

   $$\gamma''(0) = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \quad \gamma'''(0) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

Since the determinant
$$\begin{vmatrix} -3 & 0 \\ 0 & 6 \end{vmatrix} \neq 0$$
the point is a first-order inflection point, and the curve has a half-tangent vector directed by $(-3, 0)$.



1.2.3 Infinite branches

Definition 1.2.2. *A curve has an infinite branch (or an infinite arc) if at least one of the coordinates tends to infinity as $t \to t_0$ with $t_0 \in \mathbb{R} \cup \{\pm\infty\}$. The following cases are possible:*
1. If $\lim_{t \to t_0} x(t) = l \in \mathbb{R}$ and $\lim_{t \to t_0} y(t) = \pm\infty$: - The curve has a vertical asymptote with the equation $x = l$.
2. If $\lim_{t \to t_0} x(t) = \pm\infty$ and $\lim_{t \to t_0} y(t) = l \in \mathbb{R}$: - The curve has a horizontal asymptote with the equation $y = l$.
3. If $\lim_{t \to t_0} x(t) = \pm\infty$ and $\lim_{t \to t_0} y(t) = \pm\infty$, we study the limit $\lim_{t \to t_0} \frac{y(t)}{x(t)}$:

- *If $\lim_{t \to t_0} \frac{y(t)}{x(t)} = \pm\infty$, then the curve has a parabolic branch in the direction of the y-axis (Oy).*
- *If $\lim_{t \to t_0} \frac{y(t)}{x(t)} = 0$, then the curve has a parabolic branch in the direction of the x-axis (Ox).*
- *If $\lim_{t \to t_0} \frac{y(t)}{x(t)} = a \neq 0$, we study the function $y - ax$:*
  - *If $\lim_{t \to t_0} (y(t) - ax(t)) = b \in \mathbb{R}$, then the curve has an oblique asymptote with the equation $y = ax + b$.*
  - *If $\lim_{t \to t_0} (y(t) - ax(t)) = \pm\infty$, then the curve has a parabolic branch in the direction of the line with the equation $y = ax$.*

Exercise 1.2.2. *The parametric curve is defined by:*

$$\gamma(t) = (x(t), y(t)) = \left(\frac{2t}{1+t^3}, \frac{2t^2}{1+t^3}\right), \quad t \in \mathbb{R} \setminus \{-1\}$$

1. *Compute $x(1/t)$ and $y(1/t)$.*
2. *Study the curve and sketch its graph.*

solution

1. Calculation of $x(1/t)$ and $y(1/t)$

   We calculate:

$$x\left(\frac{1}{t}\right) = \frac{2 \cdot \frac{1}{t}}{1+\left(\frac{1}{t}\right)^3} = \frac{2/t}{1+1/t^3} = \frac{2}{t} \cdot \frac{t^3}{t^3+1} = \frac{2t^2}{t^3+1} = y(t).$$

$$y\left(\frac{1}{t}\right) = \frac{2\left(\frac{1}{t}\right)^2}{1+\left(\frac{1}{t}\right)^3} = \frac{2/t^2}{1+1/t^3} = \frac{2}{t^2} \cdot \frac{t^3}{t^3+1} = \frac{2t}{t^3+1} = x(t).$$

   The curve satisfies the property

$$x\left(\frac{1}{t}\right) = y(t), \quad y\left(\frac{1}{t}\right) = x(t),$$

   which means the curve is symmetric under the transformation $t \mapsto \frac{1}{t}$ that swaps its coordinates.

   Geometrically, this corresponds to a reflection symmetry with respect to the line

$$\boxed{y = x}.$$

2. Study of the curve

   - Domain and notable points

     - **Domain:** We study the curve on the interval $]-1, 1]$ then perform a symmetry with respect to the line $x = y$. .
     - **Notable points:**

$$\gamma(0) = (0, 0), \quad \gamma(1) = (1, 1).$$

   - Limits and oblique asymptote near $t = -1$

     Calculate the slope:

$$\lim_{t \to -1} \frac{y(t)}{x(t)} = \lim_{t \to -1} t = -1,$$

     so the slope of the asymptote is $m = -1$.

Calculate the intercept:

$$y(t) - (-x(t)) = \frac{2t(t+1)}{1+t^3},$$

which near $t = -1$ tends to

$$\lim_{t \to -1} y(t) + x(t) = \lim_{t \to -1} t = -\frac{2}{3},$$

Thus the oblique asymptote at $t = -1$ is

$$\boxed{y = -x - \frac{2}{3}.}$$

- Derivatives

$$x'(t) = \frac{2 - 4t^3}{(1+t^3)^2}, \quad y'(t) = \frac{4t - 2t^4}{(1+t^3)^2}.$$

- Sign of derivatives on $]-1, 1]$

- For $x'(t)$, numerator zero at $t = \sqrt[3]{\frac{1}{2}} \approx 0.7937$.

- For $y'(t)$, numerator zero at $t = 0$ in $]-1, 1]$.

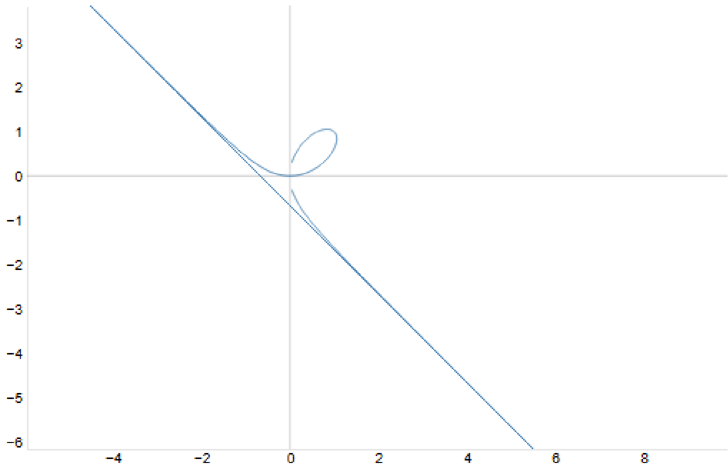
- Table of variations

t	-1		0		0.79		1
$x'(t)$	$+$	$+$		$+$	0	$-$	
			0		1.05		
$x(t)$	$-\infty$	$\nearrow$	0	$\nearrow$		$\searrow$	1
$y'(t)$	$-$	$-$	0	$+$	$+$	$+$	
	$+\infty$						
$y(t)$		$\searrow$	0	$\nearrow$		$\nearrow$	1

Note: The value $t = -1$ is excluded from the domain.

- Symmetry of the curve

The transformation $t \mapsto \frac{1}{t}$ exchanges x and y. Knowing the behavior on $]-1, 1]$ suffices to describe the entire curve on $\mathbb{R} \setminus \{-1\}$ by reflection across the line $y = x$.



$$\gamma(t) = (x(t), y(t)) = \left(\frac{2t}{1+t^3}, \frac{2t^2}{1+t^3}\right)$$

1.3 Local behavior of plane curves

$\mathbb{R}^2$ is equipped with the canonical dot product defined by:

$$(u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$$

and the canonical orientation for which the basis $(\overrightarrow{e_1}, \overrightarrow{e_2})$ is positive. The orientation will be defined by the counterclockwise (trigonometric) direction.

Let U be a vector in $\mathbb{R}^2$. Then, there exists a unique vector V such that (U, V) forms a positive basis of $\mathbb{R}^2$. We denote this vector as $U^\perp$, which is the image of U after a rotation by an angle of $\frac{\pi}{2}$:

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

If $U = (u_1, u_2)$, then $U^\perp = (-u_2, u_1)$. Moreover,

$$\det \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} = \|U\|^2 = 1 > 0$$

1.3.1 Serret-Frenet frame(moving frame)

Let (I, γ) be a regular curve. We associate to each $t \in I$ the unit tangent vector:

$$\frac{\gamma'(t)}{\|\gamma'(t)\|}$$

We denote the function:

$$\begin{array}{rcl} T : I & \to & \mathbb{R}^2 \\ t & \mapsto & \frac{\gamma'(t)}{\|\gamma'(t)\|} \end{array}$$

The normal vector is defined as:

$$\overrightarrow{N(t)} = \overrightarrow{T(t)}^\perp$$

The vector $\overrightarrow{T(t)}$ is tangent to the curve at the point $\gamma(t)$, and the vector $\overrightarrow{N(t)}$ is a vector that is normal to the curve at $\gamma(t)$.

We obtain an orthonormal moving frame, called the Serret-Frenet frame, at each point of the curve:

$$R_{\gamma(t)} = (\gamma(t), \overrightarrow{T(t)}, \overrightarrow{N(t)})$$

Example 1.3.1. *For the curve defined by the graph of the function $f : I \to \mathbb{R}$:*

$$\overrightarrow{T(t)} = \frac{1}{\sqrt{1+f'(t)^2}}\begin{pmatrix}1\\ f'(t)\end{pmatrix}$$

$$\overrightarrow{N(t)} = \frac{1}{\sqrt{1+f'(t)^2}}\begin{pmatrix}-f'(t)\\ 1\end{pmatrix}$$

•

Remark 1.3.1. *The tangent line to the curve γ at the point $\gamma(t)$ does not depend on the parameterization. However, the vector $\overrightarrow{T(t)}$ does depend on it: if the direction of the curve is reversed, then this vector will be replaced by its opposite. The Serret-Frenet frame thus depends on the direction of the parameterization as well as the orientation of the plane.*

1.3.2 Curvature

Let $\gamma : I \to \mathbb{R}^2$ be a regular curve of class C^2. We assume that this curve is parameterized by the arc length (normal parameterization). We will perform a second-order Taylor expansion of γ at s_0:

$$\gamma(s) = \gamma(s_0) + (s - s_0)\gamma'(s_0) + \frac{(s - s_0)^2}{2}\gamma''(s_0) + o((s - s_0)^2)$$

The vector $\overrightarrow{T(s)} = \gamma'(s_0)$ is tangent to the curve at $\gamma(s_0)$. If $\gamma''(s_0) \neq 0$, the Taylor expansion indicates that, to second order, the curve is "attracted" in the direction of $\gamma''(s_0)$.

In other words, the vector $\gamma''(s_0)$ gives us information about the shape of the curve in the neighborhood of $\gamma(s_0)$. Intuitively, the larger $\|\gamma''(s_0)\|$ is, the more curved the curve is.

Proposition 1.3.1. *If $\gamma : I \to \mathbb{R}^2$ is a regular parameterized curve of class C^2, then $\gamma''(s)$ is a vector collinear with $\overrightarrow{N(s)}$. In particular, there exists a continuous function $\bar{k} : I \to \mathbb{R}$ such that:*

$$\forall s \in I, \gamma''(s) = \overrightarrow{T'(s)} = \bar{k}\overrightarrow{N(s)}$$

proof :
For all $s \in I$, we have:

$$\overrightarrow{T(s)} \cdot \overrightarrow{T(s)} = \|\overrightarrow{T(s)}\|^2 = 1$$

By differentiating, we get:

$$\overrightarrow{T'(s)} \cdot \overrightarrow{T(s)} = 0$$

$T'(s)$ is orthogonal to $T(s)$, then

$$\bar{k}(s) = \overrightarrow{T'(s)} \cdot \overrightarrow{N(s)}$$

□

Proposition 1.3.2. *If $\gamma : I \to \mathbb{R}^2$ is a regular, normal parameterized curve of class C^2 and $p = \gamma(s)$ is a point on the curve:*
- The algebraic curvature $\bar{k}(s)$ of γ at p is:

$$\bar{k}(s) = \gamma''(s) \cdot \overrightarrow{N(s)} = \overrightarrow{T'(s)} \cdot \overrightarrow{N(s)}$$

- The curvature $k(s)$ of γ at p is:

$$k(s) = |\bar{k}(s)| = \|\gamma''(s)\|$$

- The point $p = \gamma(s)$ of γ is called biregular if $k(s) \neq 0$.
- A point p is called an inflection point of γ if the tangent line to γ at p crosses the curve at p in every neighborhood of p.

Figure 1.8

Example 1.3.2. *Let the parameterization* $\gamma(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right)$*,* $s \in [0, 2\pi]$*. We observe that this parameterization is normal because for all* s*, we have* $\|\gamma'(s)\| = 1$*. The curvature at the point* $\gamma(s)$ *is:*

$$k(s) = \|\gamma''(s)\| = \left\|\left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right)\right\| = \frac{1}{R}$$

Therefore, the curvature of a circle with radius R *is constant and equals* $\frac{1}{R}$*.*

•

Proposition 1.3.3. *The curvature does not depend on the parameterization, and the algebraic curvature is defined up to a sign (and depends on the direction of the curve and the orientation of the plane). More precisely, let* $\gamma_1 : I \to \mathbb{R}^2$ *be a normal parameterized curve and* $\gamma_2 : J \to \mathbb{R}^2$ *a reparameterization of* γ_1*. Then, there exists* $\varepsilon \in \{-1, 1\}$ *such that:*

$$\forall p = \gamma_1(s) = \gamma_2(u),\ k_1(s) = k_2(u) \text{ and } \bar{k_1}(s) = \varepsilon \bar{k_2}(u)$$

1.3.3 Geometric interpretation

We have seen that for a normal parameterization, the vector $\gamma''(s) = \bar{k}(s)\overrightarrow{N(s)}$ is orthogonal to $\overrightarrow{T(s)}$ and the second-order Taylor expansion of γ can be naturally rewritten in the Serret-Frenet frame:

$$\gamma(s) = \gamma(s_0) + (s - s_0)\overrightarrow{T(s_0)} + \bar{k}(s)\frac{(s - s_0)^2}{2}\overrightarrow{N(s_0)} + o((s - s_0)^2)$$

At a point with parameter s, if the curvature $\bar{k}(s)$ is strictly positive, the vector $\overrightarrow{N(s)}$ points towards the center of curvature. If the curvature is negative, then $\overrightarrow{N(s)} = \frac{1}{\bar{k}(s)}\gamma''(s)$ points in the direction opposite to the center of curvature.

Example 1.3.3. *the Serret-Frenet frame $R_i = (\gamma(s_i), \overrightarrow{T(s_i)}, \overrightarrow{N(s_i)})$ at the point with parameter s_i (with $i \in \{1, 2, 3\}$):*
- At $\gamma(s_1)$, the algebraic curvature is negative, and the vector $\overrightarrow{N(s)}$ points in the direction opposite to the center of curvature $C(s_1)$.
- At $\gamma(s_2)$, the algebraic curvature is zero, and there is no center of curvature.
- At $\gamma(s_3)$, the algebraic curvature is positive, and the vector $\overrightarrow{N(s_3)}$ points towards the center of curvature $C(s_3)$.

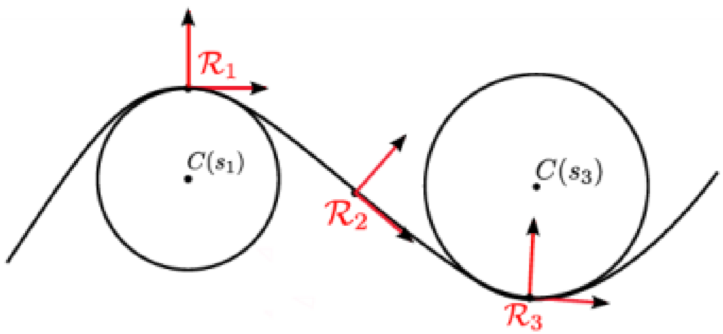


Figure 1.9

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Proposition 1.3.4. *If the curve γ has an inflection point at s_0, then the curvature at that point satisfies:*

$$k(s_0) = 0$$

Proof.
It suffices to consider the Taylor expansion once more. □

Curvature of an arbitrary curve

If we take an arbitrary parameterized curve, there is no reason for the parameterization to be normal. We can then use the following proposition to calculate the curvature:

Proposition 1.3.5. *Let $\gamma : I \to \mathbb{R}^2$ be a regular parameterized curve of class C^2. Then, for all $t \in I$, we have:*

$$\bar{k}(t) = \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} \quad \textit{and} \quad k(t) = \left| \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} \right|$$

Proof.

We denote $\tilde{\gamma} = \gamma \circ s_{t_0}^{-1}$ as the normal parameterization and set $s = s_{t_0}(t)$. The formula is:

$$(s_{t_0}^{-1})'(s) = \frac{1}{s_{t_0}'(s_{t_0}^{-1}(s))} = \frac{1}{\|\gamma'(s_{t_0}^{-1}(s))\|}$$

Thus,

$$\tilde{\gamma}'(s) = (\gamma \circ s_{t_0}^{-1})(s) = \gamma'(s_{t_0}^{-1}(s)) \cdot (s_{t_0}^{-1})'(s) = \frac{\gamma'(s_{t_0}^{-1}(s))}{\|\gamma'(s_{t_0}^{-1}(s))\|}$$

By differentiating a second time, we get:

$$\tilde{\gamma}''(s) = \frac{\gamma''(s_{t_0}^{-1}(s))}{\|\gamma'(s_{t_0}^{-1}(s))\|^2} + \alpha(s)\gamma'(s_{t_0}^{-1}(s))$$

where α is the derivative of the function $s \mapsto \frac{1}{\|\gamma'(s_{t_0}^{-1}(s))\|}$. Therefore:

$$\det(\tilde{\gamma}'(s), \tilde{\gamma}''(s)) = \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3}$$

Furthermore, we have:

$$\det(\tilde{\gamma}'(s), \tilde{\gamma}''(s)) = \det(\overrightarrow{T(s)}, \bar{k}(s)\overrightarrow{N(s)}) = \bar{k}(s)$$

which allows us to conclude.

The circle with radius $\frac{1}{k(s)}$ and tangent to the curve γ at the point $\gamma(t)$ is called the osculating circle at the point $\gamma(t)$.

□ □

Definition 1.3.1. *Let $\gamma : I \to \mathbb{R}^2$ be a regular parameterized curve of class C^2.*
- The center of curvature at $\gamma(t)$ at a point of non-zero curvature is the point:

$$C(t) = \gamma(t) + \frac{1}{\bar{k}(t)}\overrightarrow{N(t)}$$

- The osculating circle is the circle with center $C(t)$ and radius $\frac{1}{\bar{k}(s)}$.

The Serret-Frenet formulas

Definition 1.3.2 (Serret-Frenet formulas)**.** *Let $\gamma : I \to \mathbb{R}^2$ be a regular, normal parameterized curve of class C^2. Then, for all $s \in I$, we have:*

$$\begin{cases} \overrightarrow{T'}(s) = \bar{k}(s)\overrightarrow{N(s)} \\ \overrightarrow{N'}(s) = -\bar{k}(s)\overrightarrow{T(s)} \end{cases}$$

That is:

$$\frac{d}{ds}\begin{pmatrix} \overrightarrow{T(s)} \\ \overrightarrow{N(s)} \end{pmatrix} = \begin{pmatrix} 0 & \bar{k}(s) \\ -\bar{k}(s) & 0 \end{pmatrix}\begin{pmatrix} \overrightarrow{T(s)} \\ \overrightarrow{N(s)} \end{pmatrix}$$

Proof.

- The first formula has already been shown.
- We demonstrate the second formula using two methods:

First method:
Let $\overrightarrow{T}(s) = \cos(\theta(s))\overrightarrow{i} + \sin(\theta(s))\overrightarrow{j}$, then

$$\overrightarrow{N}(s) = -\sin(\theta(s))\overrightarrow{i} + \cos(\theta(s))\overrightarrow{j}$$

Differentiating, we get:

$$\frac{d}{ds}\overrightarrow{T}(s) = -\frac{d\theta}{ds}\sin(\theta(s))\overrightarrow{i} + \frac{d\theta}{ds}\cos(\theta(s))\overrightarrow{j} = \frac{d\theta}{ds}\overrightarrow{N}(s)$$

Thus,

$$\frac{d}{ds}\overrightarrow{T}(s) = \frac{d\theta}{ds}\overrightarrow{N}(s) = \bar{k}(s)\overrightarrow{N}(s)$$

therefore,

$$\frac{d\theta}{ds} = \bar{k}(s)$$

Differentiating $\overrightarrow{N}$, we get:

$$\frac{d}{ds}\overrightarrow{N}(s) = -\frac{d\theta}{ds}\cos(\theta(s))\overrightarrow{i} - \frac{d\theta}{ds}\sin(\theta(s))\overrightarrow{j} = -\bar{k}(s)\overrightarrow{T}(s)$$

Second method:
We have:

$$\forall s \in I,\ \overrightarrow{N}(s) \cdot \overrightarrow{N}(s) = 1$$

Differentiating, we get:

$$\forall s \in I,\ \overrightarrow{N}'(s) \cdot \overrightarrow{N}(s) = 0$$

Therefore, the vector $\overrightarrow{N}'(s)$ is collinear with $\overrightarrow{T}(s)$. Thus we have:

$$\forall s \in I,\ \overrightarrow{T}(s) \cdot \overrightarrow{N}(s) = 0$$

Differentiating, we get:

$$\overrightarrow{T}'(s) \cdot \overrightarrow{N}(s) + \overrightarrow{T}(s) \cdot \overrightarrow{N}'(s) = 0$$

Hence,

$$\bar{k}(s) = -\overrightarrow{N}'(s) \cdot \overrightarrow{T}(s)$$

□

□

1.4 Space curves

A space curve is a curve in $\mathbb{R}^3$ that is not planar. In this section, we consider regular space curves of class C^2 of the form:

$$\gamma : I \to \mathbb{R}^3$$

The study of the local behavior is more complicated than for plane curves. Indeed, for a regular parameterized plane curve, there is only one normal direction at each point of the curve. For a space curve, there is an entire plane that is orthogonal to the tangent vector to the curve at each point.

1.4.1 Curvature and Principal Normal

In the case of plane curves, the sign of the algebraic curvature is related to the direction of traversal of the curve and the orientation of the plane. For a space curve, it is no longer possible to have a consistent sign of curvature.

Definition 1.4.1. *Let $\gamma : I \to \mathbb{R}^3$ be a regular, normal parameterized curve of class C^2. The curvature of γ at the point $\gamma(s)$ is:*

$$k(s) = \|\gamma''(s)\|$$

Proposition 1.4.1. *The curvature of a parameterized curve $\gamma : I \to \mathbb{R}^3$ of class C^2 at the point with parameter t is:*

$$k(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

Proof.
The proof is similar to that for plane curves. □ □

Definition 1.4.2. *Let $\gamma : I \to \mathbb{R}^3$ be a regular, normal parameterized curve of class C^2.*
- A point $\gamma(s)$ is said to be biregular if $k(s) \neq 0$ (i.e., $\gamma''(s) \neq 0$).
- The principal normal of γ at a biregular point $\gamma(s)$ is given by:

$$\overrightarrow{N(s)} = \frac{1}{k(s)}\overrightarrow{T}'(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|}$$

As with plane curves, the second derivative $\gamma''(s)$ is orthogonal to $\gamma'(s)$ and indicates the direction in which the curve is curved to the second order.

Definition 1.4.3. *Let $\gamma : I \to \mathbb{R}^3$ be a regular, normal parameterized curve of class C^2 and $\gamma(s)$ a biregular point.*

- *The osculating plane at $\gamma(s)$ is the vector plane spanned by $\overrightarrow{T(s)}$ and $\overrightarrow{N(s)}$.*
- *The radius of curvature at $\gamma(s)$ is $R(s) = \frac{1}{k(s)}$.*
- *The center of curvature at $\gamma(s)$ is $C(s) = \gamma(s) + R(s)\overrightarrow{N(s)}$.*
- *The evolute of γ is the set of centers of curvature.*
- *The osculating sphere is the sphere with center $C(s)$ and radius $R(s)$.*

Remark 1.4.1.
It can be shown that these definitions do not depend on the parametrization of the geometric curve.

1.4.2 Binormal and Serret-Frenet Frame

The Serret-Frenet frame is an orthonormal frame that varies along a parameterized curve. The vector $\overrightarrow{T(s)}$ is a unit tangent to the curve, and the principal normal $\overrightarrow{N(s)}$ is orthogonal to $\overrightarrow{T(s)}$. It is natural to complete this into an orthonormal basis by defining:

$$\overrightarrow{B(s)} = \overrightarrow{T(s)} \wedge \overrightarrow{N(s)}$$

where $\wedge$ is the cross product defined by:

$$(u_1, u_2, u_3) \wedge (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

Definition 1.4.4. *Let $\gamma : I \to \mathbb{R}^3$ be a regular parameterized curve and $\gamma(s)$ a biregular point.*
- The binormal vector to γ at the point $\gamma(s)$ is:

$$\overrightarrow{B(s)} = \overrightarrow{T(s)} \times \overrightarrow{N(s)}$$

- The Serret-Frenet frame of γ at the point $\gamma(s)$ is the direct orthonormal frame:

$$(\gamma(s), \overrightarrow{T(s)}, \overrightarrow{N(s)}, \overrightarrow{B(s)})$$

1.4.3 Torsion of a Curve

We will now study the torsion of a curve, in other words, how the Serret-Frenet frame "twists" around the tangent line to the curve. To measure this, we need the derivative of the binormal vector $\overrightarrow{B}$. Since $\overrightarrow{B}$ involves the second derivative of the parameterization, we must consider parameterizations of class C^3.

Here, we consider a curve $\gamma : I \to \mathbb{R}^3$ that is a normal parameterized curve of class C^3 and $\gamma(s)$ a biregular point.

Proposition 1.4.2. *The vector $\overrightarrow{B'}(s)$ is collinear with $\overrightarrow{N}(s)$. In particular, there exists a continuous function $\tau : I \to \mathbb{R}$ such that:*

$$\forall s \in I, \overrightarrow{B'}(s) = \tau(s)\overrightarrow{N}(s)$$

Proof.

For all $s \in I$, we have:

$$\overrightarrow{B}(s) \cdot \overrightarrow{B}(s) = \|\overrightarrow{B}(s)\| = 1$$

By differentiating, we get:

$$\forall s \in I, \overrightarrow{B'}(s) \cdot \overrightarrow{B'}(s) = 0$$

Similarly, for all $s \in I$, we have:

$$\overrightarrow{B}(s) \cdot \overrightarrow{T}(s) = 0$$

By differentiating, we get:

$$\forall s \in I, \overrightarrow{B'}(s) \cdot \overrightarrow{T}(s) + \overrightarrow{B}(s) \cdot \overrightarrow{T'}(s) = 0 \quad (\text{where} \overrightarrow{T'}(s) = \bar{k}(s)\overrightarrow{N}(s))$$

Thus:

$$\overrightarrow{B'}(s) \cdot \overrightarrow{T}(s) = 0$$

Therefore, the vector $\overrightarrow{B'}(s)$ is orthogonal to both $\overrightarrow{T}(s)$ and $\overrightarrow{B}(s)$. It is then collinear with $\overrightarrow{N}(s)$. We define:

$$\tau(s) = \overrightarrow{B'}(s) \cdot \overrightarrow{N}(s)$$

□

Definition 1.4.5. *The torsion $\tau(s)$ of a curve parameterized by arc length γ at a point $\gamma(s)$ is:*

$$\tau(s) = \overrightarrow{B'}(s) \cdot \overrightarrow{N}(s)$$

Proposition 1.4.3. *The torsion $\tau(s)$ of a curve for all $s \in I$ is given by:*

$$\tau(s) = -\frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\|\gamma''(s)\|^2}$$

Proof. For all $s \in I$, we have:

$$\begin{aligned}
\tau(s) &= \vec{N}(s) \cdot \vec{B}'(s) \\
\vec{B}'(s) &= \vec{N}(s) \cdot \left(\vec{T}'(s) \wedge \vec{N}(s) + \vec{T}(s) \wedge \vec{N}'(s)\right) \\
&= \vec{N}(s) \cdot \left(\vec{T}(s) \wedge \vec{N}'(s)\right) \\
&= \det\left(\vec{N}(s), \vec{T}(s), \vec{N}'(s)\right) \\
&= -\det\left(\vec{T}(s), \vec{N}(s), \vec{N}'(s)\right) \\
&= -\det\left(\gamma'(s), R(s)\gamma''(s), R'(s)\gamma''(s) + R(s)\gamma'''(s)\right) \\
&= -R^2(s)\det\left(\gamma'(s), \gamma''(s), \gamma'''(s)\right) \\
&= -\frac{\det\left(\gamma'(s), \gamma''(s), \gamma'''(s)\right)}{\|\gamma'(s) \wedge \gamma''(s)\|^2}
\end{aligned}$$

□

□

Torsion for a General Parameterization

Proposition 1.4.4. *The torsion of a parameterized curve $\gamma : I \to \mathbb{R}^3$ of class C^3 at a point $\gamma(t)$ is given by:*

$$\tau(t) = -\frac{\det(\gamma'(t), \gamma''(t), \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

Geometric Interpretation of Torsion

Torsion measures how the vector $\overrightarrow{B}$ "twists". A geometric interpretation of torsion is given by the following proposition

Proposition 1.4.5. *Let $\gamma : I \to \mathbb{R}^3$ be a regular parameterized curve of class C^3 whose all points are biregular. Then: The curve γ is planar if and only if* $\forall t \in I, \tau(t) = 0$

1.4.4 Serret-Frenet formulas

Proposition 1.4.6 (Serret-Frenet formulas)**.** *Let $\gamma : I \to \mathbb{R}^3$ be a regular parameterized curve of class C^3 and $\gamma(t)$ a biregular point.*
Then we have:

$$\begin{cases} \overrightarrow{T'}(s) = k(s)\overrightarrow{N}(s) \\ \overrightarrow{N'}(s) = -k(s)\overrightarrow{T}(s) - \tau(s)\overrightarrow{B}(s) \\ \overrightarrow{B'}(s) = \tau(s)\overrightarrow{N}(s) \end{cases}$$

That is:

$$\begin{pmatrix} \overrightarrow{T'}(s) \\ \overrightarrow{N'}(s) \\ \overrightarrow{B'}(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{T}(s) \\ \overrightarrow{N}(s) \\ \overrightarrow{B}(s) \end{pmatrix}$$

Proof.

We only need to show the second formula. For this, we will calculate the coordinates of $\overrightarrow{N}(s)$ in the Serret-Frenet frame.

For all $s \in I$, we have:

$$\overrightarrow{N}(s) \cdot \overrightarrow{N}(s) = 1$$

Differentiating, we get:

$$\overrightarrow{N'}(s) \cdot \overrightarrow{N}(s) = 0$$

And for all $s \in I$, we have:

$$\overrightarrow{B}(s) \cdot \overrightarrow{N}(s) = 0$$

Therefore:

$$\overrightarrow{N'}(s) \cdot \overrightarrow{B}(s) = -\overrightarrow{N}(s) \cdot \overrightarrow{B'}(s) = -\overrightarrow{N}(s) \cdot (\tau(s)\overrightarrow{N}(s)) = -\tau(s)$$

Similarly, for all $s \in I$, we have:

$$\overrightarrow{N}(s) \cdot \overrightarrow{T}(s) = 0$$

Differentiating, we get:

$$\overrightarrow{N'}(s) \cdot \overrightarrow{T}(s) = -\overrightarrow{N}(s) \cdot \overrightarrow{T'}(s) = -\overrightarrow{N}(s) \cdot (k(s)\overrightarrow{N}(s)) = -k(s)$$

□ □

Exercise 1.4.1.
Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the parametrized curve defined by:

$$\begin{cases} x(t) = 1 + t \\ y(t) = -t^3 \\ z(t) = 1 + t^3 \end{cases}$$

1. *Show that this representation is regular. Find a system of Cartesian equations for the affine line in $\mathbb{R}^3$ tangent to the curve γ at the point $\gamma(1)$.*
2. *Find an equation of the affine plane in $\mathbb{R}^3$ normal to the curve at the point $\gamma(1)$.*

Solution :

1. We compute the derivative of the curve:

$$\gamma'(t) = \big(1,\ -3t^2,\ 3t^2\big)$$

Since $x'(t) = 1 \neq 0$, the vector $\gamma'(t)$ never vanishes, so the curve is regular.

Evaluate at $t = 1$:

$$\gamma(1) = (2,\ -1,\ 2), \quad \vec{u} = \gamma'(1) = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$$

A point $M = (x, y, z)$ lies on the tangent line (Δ) at $M_0 = \gamma(1)$ if:

$$\begin{cases} x - 2 = \lambda \\ y + 1 = -3\lambda \\ z - 2 = 3\lambda \end{cases}$$

Eliminating λ yields the Cartesian equations:

$$\begin{cases} 3x - z = 4 \\ y + z = 1 \end{cases}$$

2. A point $M = (x, y, z)$ lies on the affine plane normal to the curve at $\gamma(1)$ if:

$$(x - 2, y + 1, z - 2) \cdot (1, -3, 3) = 0$$

$$\Rightarrow x - 3y + 3z = 11$$

Thus, the equation of the normal plane is:

$$x - 3y + 3z = 11$$

Exercise 1.4.2.
Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be the parametrized curve defined by:

$$\begin{cases} x(s) = \dfrac{1}{2}\sin s \\ y(s) = \left(\sin\left(\dfrac{s}{2}\right)\right)^2 \\ z(s) = \dfrac{\sqrt{3}}{2}s \end{cases}$$

1. *Show that γ is a regular curve parametrized by arc length.*
2. *For each $s \in \mathbb{R}$, compute the curvature $K(s)$ and the torsion $\tau(s)$ of the curve γ.*
3. *For each $s \in \mathbb{R}$, explicitly give the Frenet frame $\left(\vec{T}(s), \vec{N}(s), \vec{B}(s)\right)$.*

solution

1. Regularity and arc-length parametrization:

   We compute the tangent vector to γ:
   $$\vec{T}(s) = \dot{\gamma}(s) = \frac{1}{2}\left(\cos s,\ \sin s,\ \sqrt{3}\right)$$
   Its norm is:
   $$|\dot{\gamma}(s)| = \frac{1}{2}\sqrt{(\cos s)^2 + (\sin s)^2 + 3} = \frac{1}{2}\sqrt{1+3} = 1$$
   Thus, γ is regular and parametrized by arc length.

2. Curvature and Torsion:

   The second derivative is:
   $$\ddot{\gamma}(s) = \frac{1}{2}(-\sin s,\ \cos s,\ 0)$$
   The curvature is:
   $$K(s) = \|\ddot{\gamma}(s)\| = \frac{1}{2}$$
   The unit normal vector is:
   $$\vec{N}(s) = \frac{\ddot{\gamma}(s)}{K(s)} = (-\sin s,\ \cos s,\ 0)$$

   Compute the binormal vector:
   $$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) = -\frac{\sqrt{3}}{2}\begin{pmatrix} \cos s \\ \sin s \\ -\dfrac{\sqrt{3}}{3} \end{pmatrix}$$

   Differentiate:
   $$\vec{B}\,'(s) = -\frac{\sqrt{3}}{2}(-\sin s,\ \cos s,\ 0) = -\frac{\sqrt{3}}{2}\vec{N}(s)$$
   Hence, the torsion is:
   $$\tau(s) = -\frac{\sqrt{3}}{2}$$

3. Frenet Frame:

$$\vec{T}(s) = \frac{1}{2}\begin{pmatrix} \cos s \\ \sin s \\ \sqrt{3} \end{pmatrix} \quad \vec{N}(s) = \begin{pmatrix} -\sin s \\ \cos s \\ 0 \end{pmatrix} \quad \vec{B}(s) = -\frac{\sqrt{3}}{2}\begin{pmatrix} \cos s \\ \sin s \\ -\frac{\sqrt{3}}{3} \end{pmatrix}$$

Exercise 1.4.3.
We consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by:

$$\gamma(t) = \left(t,\ \frac{3}{2}t^2,\ \frac{3}{2}t^3\right) \quad \textit{for all } t \in \mathbb{R}.$$

1. *Find two Cartesian equations (in x, y, z) describing the support Γ of γ.*
2. *Determine the regular points of γ.*
3. *Find an arc-length parameter $s(t)$. Compute the length of γ between $\gamma(0)$ and $\gamma(10)$.*
4. *Compute the curvature of γ. Determine the bi-regular points of γ.*
5. *Compute the torsion of γ.*
6. *Determine the tangent, normal, and binormal vectors of γ.*

solution

1. Let
$$x = t, \quad y = \frac{3}{2}t^2, \quad z = \frac{3}{2}t^3.$$
Then the support Γ is given by the Cartesian equations
$$y = \frac{3}{2}x^2 \quad \text{and} \quad z = xy \quad \left(\text{or equivalently } z = \frac{3}{2}x^3\right), \quad \text{for all } x, z \in \mathbb{R} \text{ and } y \in \mathbb{R}^+.$$

2. The derivative of γ is
$$\gamma'(t) = (1, 3t, \frac{9}{2}t^2) \neq (0,0,0) \quad \text{for all } t \in \mathbb{R},$$
so γ is regular at every point.

3. The arc length function is
$$s(t) = \int_0^t \sqrt{1 + \frac{9}{2}u^2}\, du = t + \frac{3}{2}t^3.$$
Hence, the length of γ between $\gamma(0)$ and $\gamma(10)$ is
$$L(\gamma) = |s(10) - s(0)| = 1510.$$

4. The curvature is
$$k(t) = \frac{\|\gamma'(t) \wedge \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$
Since
$$\gamma''(t) = (0, 3, 9t)\,,$$
we have
$$\gamma'(t) \wedge \gamma''(t) = \left(\frac{27}{2}t^2, -9t, 3\right),$$
and
$$\|\gamma'(t) \wedge \gamma''(t)\|^2 = \left(\frac{27}{2}t^2\right)^2 + (-9t)^2 + 3^2 = \frac{729}{4}t^4 + 81t^2 + 9 = 9\left(\frac{81}{4}t^4 + 9t^2 + 1\right).$$
This simplifies to
$$\|\gamma'(t) \wedge \gamma''(t)\| = 3\left(\frac{9}{2}t^2 + 1\right),$$
and
$$\|\gamma'(t)\|^2 = 1 + \frac{9}{2}t^2,$$
so
$$k(t) = \frac{3\left(\frac{9}{2}t^2 + 1\right)}{\left(1 + \frac{9}{2}t^2\right)^{3/2}} = \frac{3}{\sqrt{2 + 9t^2}} \neq 0 \quad \text{for all } t \in \mathbb{R}.$$
Thus, γ is biregular everywhere.

5. The torsion is given by
$$\tau(t) = \frac{[\gamma'(t), \gamma''(t), \gamma'''(t)]}{\|\gamma'(t) \wedge \gamma''(t)\|^2},$$
where denotes the scalar triple product. Since
$$\gamma'''(t) = (0, 0, 9),$$
the determinant (scalar triple product) is
$$\det(\gamma'(t), \gamma''(t), \gamma'''(t)) = 27,$$
and the torsion is
$$\tau(t) = \frac{27}{9\left(\frac{9}{2}t^2 + 1\right)^2} = \frac{12}{(9t^2 + 2)^2}.$$

6. The Frenet frame vectors are given by:
$$T(s) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{\sqrt{2 + 9t^2}}(2, 6t, 9t^2),$$
$$N(s) = \frac{T'(s)}{\|T'(s)\|} = \frac{1}{\sqrt{2 + 9t^2}}(-6t, 2 - 9t^2, 6t),$$
$$B(s) = T(s) \wedge N(s) = \frac{1}{\sqrt{2 + 9t^2}}(9t^2, -6t, 2).$$

1.4.5 Implicitly Defined Curves

An implicitly defined curve is the zero set of a function $F : \mathbb{R}^2 \to \mathbb{R}$ in the plane, or of two functions $F, G : \mathbb{R}^3 \to \mathbb{R}$ in the ambient space. This corresponds to the notion of a curve as a submanifold of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Definition 1.4.6. *A plane curve is a subset of $\mathbb{R}^2$ of the form*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0, \quad x \in A, \quad y \in B\},$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a function. The curve is said to be regular *at a point (x_0, y_0) if F is differentiable at (x_0, y_0) and the gradient of F at (x_0, y_0) is nonzero:*

$$\nabla F(x_0, y_0) = \left(\frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0)\right) \neq (0, 0).$$

In this case, $\nabla F(x_0, y_0)$ is a normal vector to Γ at (x_0, y_0).

Example 1.4.1.

- *The circle*
  $$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$$
  is regular everywhere, since the point $(0, 0)$, where the gradient of the function
  $$f(x, y) = x^2 + y^2 - r^2$$
  vanishes, does not belong to the curve.
- *The arc of the circle contained in the first quadrant is the curve*
  $$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2, \quad x \in [0, 1], \quad y \in [0, 1]\}.$$
- *The graph of a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is the curve*
  $$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$$
  which is regular everywhere that f is differentiable.
- *The curve*
  $$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 + x^2\}$$
  is singular at the origin.

•

Definition 1.4.7. *A* space curve *is a subset of* $\mathbb{R}^3$ *of the form:*

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0, \quad G(x, y, z) = 0, \quad x \in A,\, y \in B,\, z \in C\}$$

where $F, G : \mathbb{R}^3 \to \mathbb{R}$ *are two functions. The curve is said to be* regular *at a point* (x_0, y_0, z_0) *if* F *and* G *are differentiable at* (x_0, y_0, z_0) *and the two gradients* $\nabla F(x_0, y_0, z_0)$ *and* $\nabla G(x_0, y_0, z_0)$ *are linearly independent, which happens if and only if:*

$$\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) \neq \mathbf{0}.$$

In this case, they generate the normal plane to Γ *at* (x_0, y_0, z_0).

Example 1.4.2.

- ***The circle***

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2, \quad 3x + y - z = 0\}$$

is a regular curve everywhere.

- ***The Viniani window***

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad x + z^2 = 1\}$$

is singular at the point $(1, 0, 0)$, *because this point cancels the cross product of the gradients of the functions*

$$F(x, y, z) = x^2 + y^2 + z^2 - 1, \quad G(x, y, z) = x + z^2 - 1.$$

The support of a parametrized curve can always be described implicitly: it suffices to find the Cartesian equation that constrains the coordinates of its points. The converse is false: there exist implicitly defined curves that do not admit a global parametrization.

Example 1.4.3.

- ***The branch of a hyperbola***

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1, \quad x > 0\}$$

admits the parametrization

$$\gamma(t) = (\cosh t, \sinh t), \quad t \in \mathbb{R}.$$

- ***The curve***

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^6 = 1\}$$

does not admit a global parametrization.

Theorem 1.4.1.
Every curve can be parametrized locally around a regular point, that is, in an open neighborhood of the point.

Proof.
This is a direct consequence of the implicit function theorem. Consider the case of plane curves. Let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$$

be a curve regular at (x_0, y_0), i.e., such that $\nabla F(x_0, y_0) \neq (0, 0)$, and suppose that $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.

Then there exist an open set U containing x_0, an open set V containing y_0, and a differentiable function $\varphi : U \to V$ such that $\varphi(x_0) = y_0$ with the following property: For every $(x, y) \in U \times V$, we have

$$F(x, y) = 0 \iff y = \varphi(x).$$

Moreover,

$$\varphi'(x_0) = -\frac{\frac{\partial F}{\partial x}(x_0, \varphi(x_0))}{\frac{\partial F}{\partial y}(x_0, \varphi(x_0))}.$$

Therefore, the map

$$\gamma(t) = (t, \varphi(t)) \in \mathbb{R}^2, \quad \forall t \in U,$$

is a parametrization of Γ in a neighborhood of (x_0, y_0), regular at this point. $\square$

Example 1.4.4.
The curve

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^6 = 1\}$$

around the point $(0, 1)$, *admits the parametrization*

$$\gamma(t) = \left(t, \sqrt[6]{1 - t^4}\right), \quad t \in (-1, 1).$$

Remark 1.4.2.
If γ is a regular parametrized curve, its support Γ is a regular curve. The converse is false: it is possible to parametrize a regular support in a non-regular way.

Example 1.4.5.
The x-axis

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$$

is an implicit regular curve, because the gradient of the function $F(x, y) = y$ is the constant nonzero vector $\nabla F(x, y) = (0, 1)$. However, this axis can be parametrized in a non-regular way, for example by

$$\gamma(t) = (t^3, 0), \quad t \in \mathbb{R},$$

which is not regular at $t = 0$.

Chapter 2

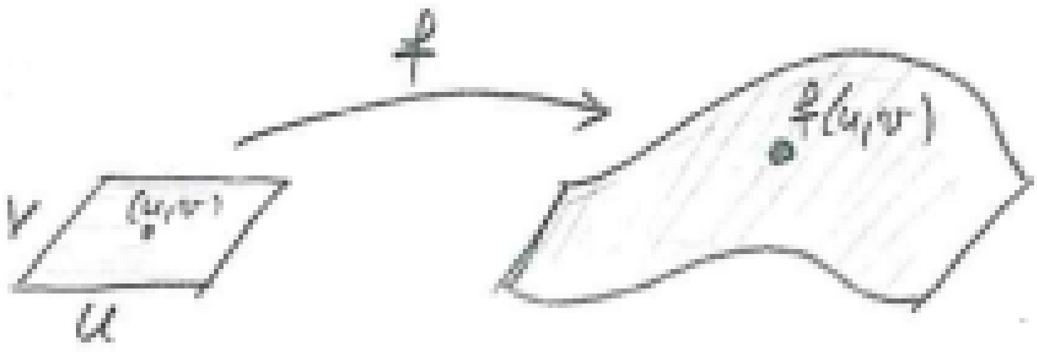
Surfaces

Definition 2.0.1. *A parameterized surface of class C^k (with $k \geq 0$) is a subset of $\mathbb{R}^3$ of the form*

$$S = \big\{ f(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \in \mathbb{R}^3 \;\big|\; u \in U \subset \mathbb{R},\; v \in V \subset \mathbb{R} \big\} = f(U \times V)$$

where $U, V \subset \mathbb{R}$ are two intervals and the map $f : U \times V \to \mathbb{R}^3$ is of class C^k, that is, the functions $x, y, z : U \times V \to \mathbb{R}$ are of class C^k. If the class C^k is not specified, we assume that the surface is smooth, that is, of class C^∞.
We call:

- ***parameterization*** *the map $f : U \times V \to \mathbb{R}^3$.*
- ***parameters*** *the variables $u \in U$ and $v \in V$.*
- ***(geometric) support*** *of f its image supp $f = f(U \times V) \subset \mathbb{R}^3$.*

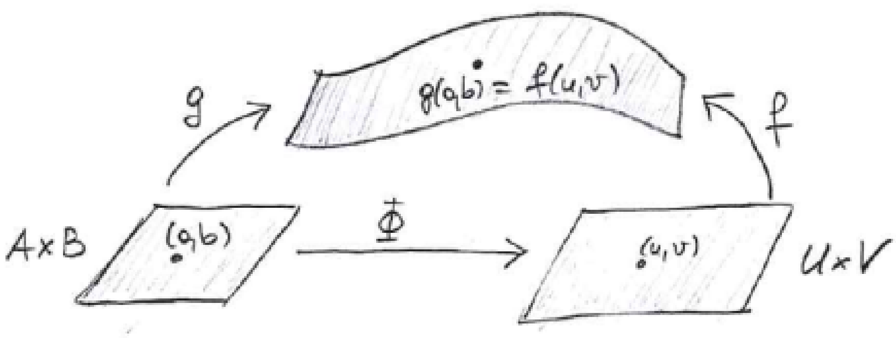


Example 2.0.1.

- ***Cylinder*** $f(u,v) = \big(r\cos u,\ r\sin u,\ v\big) \in \mathbb{R}^3$ *with* $u, v \in \mathbb{R}$
- ***Helicoid*** $f(u,v) = \big(u\cos v,\ u\sin v,\ v\big) \in \mathbb{R}^3$ *with* $u, v \in \mathbb{R}$

Cylinder ***Helicoid***

Definition 2.0.2. *Let* $S \subset \mathbb{R}^3$ *be the surface parametrized by* $f : U \times V \to \mathbb{R}^3$. *A reparametrization of* S *is a new parametrization* $g : A \times B \to \mathbb{R}^3$ *of* S *obtained by composing* f *with a diffeomorphism* $\Phi : A \times B \to U \times V$, *i.e., such that* $g = f \circ \Phi$. *The new parameters are* $(a, b) = \Phi^{-1}(u, v)$.



Example 2.0.2.

1. *The map* $\Phi(a,b) = (a^2, b)$ *is not invertible.*
2. *The map* $\Phi(a,b) = (a^3, b)$ *is invertible, but its inverse* $\Phi^{-1}(u,v) = (\sqrt[3]{u}, v)$ *is not differentiable at* $u = 0$.
3. *The map* $\Phi(a,b) = (e^{a+b}, a-b)$ *is a diffeomorphism, with inverse:*

$$\Phi^{-1}(u,v) = \left(\frac{1}{2}(\ln u + v), \frac{1}{2}(\ln u - v)\right).$$

4. *Let* S *be the surface parametrized by:*

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v) = \left(e^v, (u-v)e^{-v}, u-v\right).$$

The map $\Phi^{-1} : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}_+^*$, *given by:*

$$(u,v) \mapsto \Phi^{-1}(u,v) = (u-v, e^v) = (a,b),$$

is a diffeomorphism with inverse

$$\Phi : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}^2, \quad (a,b) \mapsto \Phi(a,b) = (a + \ln b, \ln b).$$

Thus, the map

$$g = f \circ \Phi : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}^3, \quad (a,b) \mapsto g(a,b) = f(a + \ln b, \ln b) = \left(b, \frac{a}{b}, a\right)$$

is a reparametrization of the surface.

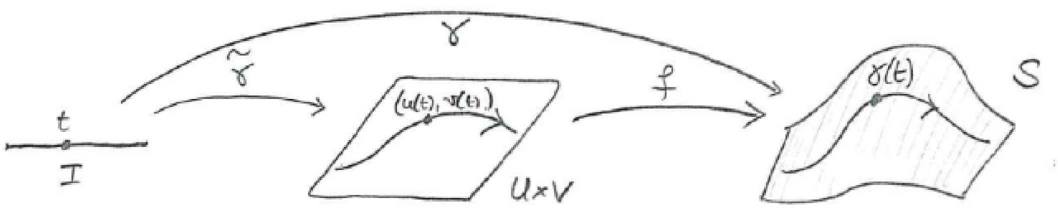
•

2.0.1 Curves on a surface

Definition 2.0.3. *Let S be a surface parametrized by $f : U \times V \to \mathbb{R}^3$. A curve parametrized on S is a curve parametrized by $\gamma : I \to S$ obtained by composing f with the parametrization $\tilde{\gamma}(t) = (u(t), v(t))$, such that*

$$\gamma(t) = f(u(t), v(t)),$$

for all $t \in I$.



Curves on a surface

Example 2.0.3.

For fixed $a \in \mathbb{R}$, the helix $\gamma(t) = (a\cos t, a\sin t, t)$ is a space curve that can be seen as:

- *a curve lying on the cylinder parametrized by*

$$f(u, v) = (a\cos u, a\sin u, v) \in \mathbb{R}^3,$$

  where the parameters u and v are themselves parametrized by

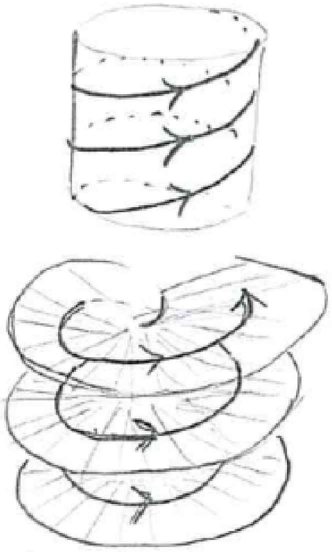
$$u(t) = t \quad \text{and} \quad v(t) = t.$$

- *a curve lying on the helicoid parametrized by*

$$f(u, v) = (u\cos v, u\sin v, v) \in \mathbb{R}^3,$$

  where the parameters u and v are themselves parametrized by

$$u(t) = a \quad \text{and} \quad v(t) = t.$$



2.0.2 Regular surfaces

Definition 2.0.4. *We say that the surface S, parametrized by $f : U \times V \to \mathbb{R}^3$, is:*

- ***Regular*** *at $(u_0, v_0) \in U \times V$ if the partial derivative vectors of f at (u_0, v_0), denoted by*

$$\frac{\partial f(u_0, v_0)}{\partial u} = \partial_u f(u_0, v_0) = f_u(u_0, v_0), \quad \frac{\partial f(u_0, v_0)}{\partial v} = \partial_v f(u_0, v_0) = f_v(u_0, v_0),$$

are linearly independent (and hence non-zero), i.e., their cross product is non-zero:

$$\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0) = f_u(u_0, v_0) \wedge f_v(u_0, v_0) \neq 0.$$

- ***Singular*** *at $(u_0, v_0) \in U \times V$ if the vectors $\partial_u f(u_0, v_0)$ and $\partial_v f(u_0, v_0)$ are linearly dependent, i.e.,*

$$\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0) = f_u(u_0, v_0) \wedge f_v(u_0, v_0) = 0.$$

Example 2.0.4.

- *The surface parameterized by $f(u, v) = (u^2, v^2, uv) \in \mathbb{R}^3$, with $u, v \in \mathbb{R}$, is singular at $(0, 0)$, because*

$$\begin{cases} \partial_u f(u, v) & = (2u, 0, v) \\ \partial_v f(u, v) & = (0, 2v, u) \end{cases} \quad \Rightarrow \quad (\partial_u f \wedge \partial_v f)(u, v) = (-2v^2, -2u^2, 4uv)$$

Thus, the vector $(\partial_u f \wedge \partial_v f)(u, v)$ vanishes at $(0, 0)$.

- *The graph of a function $h : \mathbb{R}^2 \to \mathbb{R}$ defines a surface S parameterized by $f(u, v) = \big(u, v, h(u, v)\big)$, with $(u, v) \in D_h \subset \mathbb{R}^2$. If h is of class $\mathcal{C}^1$, the surface S is regular everywhere :*

$$\begin{cases} \partial_u f(u, v) & = (1, 0, \partial_u h(u, v)) \\ \partial_v f(u, v) & = (0, 1, \partial_v h(u, v)) \end{cases} \quad \Rightarrow \quad (\partial_u f \wedge \partial_v f)(u, v) = (-\partial_u h(u, v), -\partial_v h(u, v), 1) \neq \vec{0}.$$

Definition 2.0.5.

Let S be a surface parameterized by $f : U \times V \to \mathbb{R}^3$, and let $P_0 = f(u_0, v_0)$ be a regular point of S. We define:

- *The **tangent plane** to S at the point P_0 is the plane spanned by the vectors $\partial_u f(u_0, v_0)$ and $\partial_v f(u_0, v_0)$, passing through P_0:*

$$\begin{aligned} T_{P_0}S &= P_0 + \mathrm{Vect}\big(\partial_u f(u_0, v_0), \partial_v f(u_0, v_0)\big) \\ &= f(u_0, v_0) + \lambda\, \partial_u f(u_0, v_0) + \mu\, \partial_v f(u_0, v_0) \end{aligned}$$

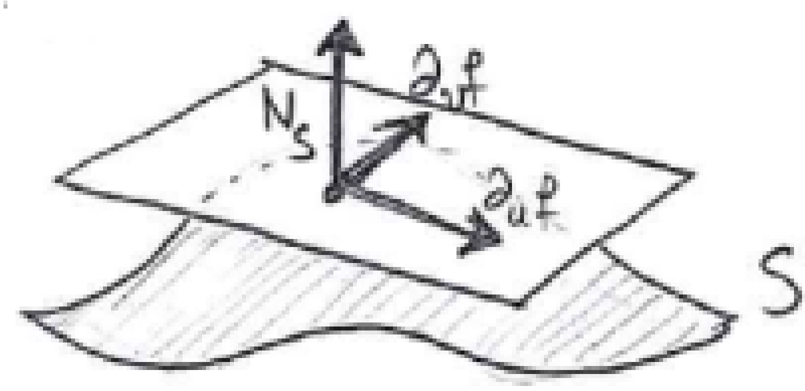
- *The **normal vector** (unit) to S at P_0 is the vector*

$$N_S(u_0, v_0) = \frac{\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)}{\|\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)\|}.$$

The tangent plane to S at (u_0, v_0) contains the tangent line to every regular curve on S passing through $f(u_0, v_0)$. Indeed, if $\gamma(t) = f(u(t), v(t))$ is such a curve, and $(u_0, v_0) = (u(t_0), v(t_0))$, then

$$\gamma'(t_0) = \frac{\partial f}{\partial x}(u_0, v_0)u'(t_0) + \frac{\partial f}{\partial y}(u_0, v_0)v'(t_0) \in \mathrm{Vect}\left(\partial_u f(u_0, v_0), \partial_v f(u_0, v_0)\right).$$

By definition, these three vectors $(\partial_u f(u_0, v_0), \quad \partial_v f(u_0, v_0), \quad N_S(u_0, v_0))$ form a direct basis of the space above the point $f(u_0, v_0)$ of the surface (that is, a frame).



Remark 2.0.1.
This basis $\partial_u f(u_0, v_0), \quad \partial_v f(u_0, v_0), \quad N_S(u_0, v_0)$ is neither orthogonal nor normalized.

2.0.3 Surfaces defined by the equation $z = f(x, y)$

For these surfaces, we consider a function f of two variables $(x, y) \in D \subset \mathbb{R}^2$, of class $\mathcal{C}^k$ with values in $\mathbb{R}$, and we define the surface S as the set of points $M(x, y, z)$ where $z = f(x, y)$. Thus, this is a particular case of parametrized surfaces:

$$\begin{array}{rcl} F : \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x, y) & \mapsto & (x, y, f(x, y)) \end{array}$$

Regular points

Theorem 2.0.1.
All points of a surface with equation $z = f(x, y)$, where f is of class $\mathcal{C}^k$ $(k \geq 1)$, are regular.

Proof. exercise □

Equation of the tangent plane

Let $P = (x, y, z)$ be any point of the tangent plane to S at M_0. To find an equation of the tangent plane at M_0 to S, it suffices that the scalar product of $\overrightarrow{M_0P}$ and $\overrightarrow{N_0}$ (the normal vector to S at M_0 is zero, i.e.,

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - z = f(x_0, y_0, z_0) = 0$$

Exercise 2.0.1. *The paraboloid of revolution is given by $z = f(x, y) = x^2 + y^2$*

- *Calculate the equation of the tangent plane to this surface at a point $M_0 = (x_0, y_0, z_0)$. Then calculate it at the point $O = (0, 0, 0)$.*

2.0.4 Implicitly Defined Surfaces

Definition 2.0.6.
An implicitly defined surface is the zero set of a function $F : \mathbb{R}^3 \to \mathbb{R}$, that is:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0, \quad x \in A,\ y \in B,\ z \in C\}.$$

The surface is regular at the point (x_0, y_0, z_0) if F is differentiable at (x_0, y_0, z_0) and the gradient $\nabla F(x_0, y_0, z_0)$ is nonzero. In this case, the gradient defines a normal line to S at (x_0, y_0, z_0), and the tangent plane is the plane orthogonal to $\nabla F(x_0, y_0, z_0)$ passing through (x_0, y_0, z_0).

Example 2.0.5.

- *The surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = e^z\}$ is regular everywhere.*
- *The surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 + 1\}$ is regular everywhere.*
- *The surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ is singular at $(0, 0, 0)$.*

2.0.5 Surfaces of Revolution and Ruled Surfaces

Definition 2.0.7.
A surface of revolution is a surface S for which there exists a parametrization of the form

$$f : U \times [0, 2\pi[\to \mathbb{R}^3, \quad (u, \varphi) \mapsto f(u, \varphi) = R_{\varphi}^{\vec{l}}\big(\alpha(u)\big),$$

where:

- $\alpha : U \to \mathbb{R}^2$ *is a* $\mathcal{C}^1$ *curve called the* meridian *of* S*,*
- $R_{\varphi}^{\vec{l}}$ *is the rotation by an angle* φ *around a line directed by* $\vec{l}$*, contained in the plane of the meridian* α*, called the* axis of revolution *of* S*.*

We then have $S = \bigcup_{\varphi \in [0,2\pi[} R_{\varphi}^{\vec{l}}\Gamma$*, where* $\Gamma = \alpha(U)$ *is the support of* α*.*
We call:

- *the* meridians *of* S *the curves* Γ_{φ_0} *on* S *at fixed angle* φ_0*:*

$$\Gamma_{\varphi_0} = \{\alpha(u) = f(u, \varphi_0) \mid u \in U\};$$

- *the* parallels *of* S *the curves* Γ_{u_0} *on* S *at fixed height* u_0*;*

$$\Gamma_{u_0} = \{\beta(\varphi) = f(u_0, \varphi) \mid \varphi \in [0, 2\pi[\} = \textit{circle}.$$

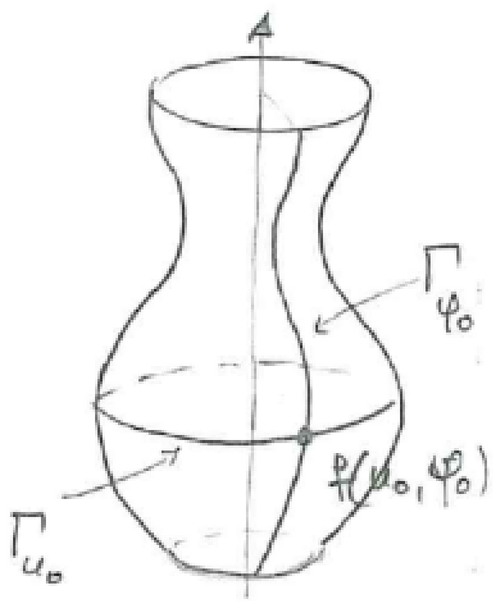
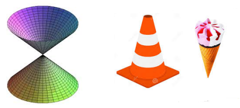


Figure 2.7

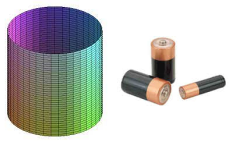
Example 2.0.6.

- *The cone* $x^2 + y^2 = z^2$



The cone

- *The cylinder* $x^2 + y^2 = r^2$



The cylinder

- *The sphere* $x^2 + y^2 + z^2 = r^2$

The sphere

- *The torus*

The torus

- *The one-sheet hyperboloid* $x^2 + y^2 - z^2 = 1$.

  If we denote by S *the hyperboloid, its intersection with the plane* $\pi = \{y = 0\}$ *gives two possible meridians:*

$$S \cap \pi = \{(x, 0, z) \mid x^2 - z^2 = 1\} = \alpha(\mathbb{R}) \cup \tilde{\alpha}(\mathbb{R}),$$

  where

$$\alpha(u) = (\cosh u, 0, \sinh u) \quad \textit{and} \quad \tilde{\alpha}(u) = (-\cosh u, 0, \sinh u).$$

  One of them is sufficient to cover S *by rotation: if*

$$R_\varphi^{\vec{z}} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

  then

$$S = \bigcup_{\varphi \in [0, 2\pi[} R_\varphi^{\vec{z}} \alpha(\mathbb{R}) = \{f(u, \varphi) = (\cos\varphi \cosh u, \sin\varphi \cosh u, \sinh u) \mid u \in \mathbb{R}, \varphi \in [0, 2\pi[\}.$$

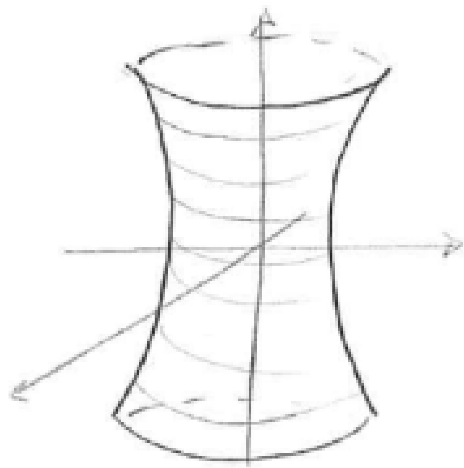


Figure 2.9

Ruled surfaces

Definition 2.0.8.

A surface S is called a ruled surface if there exists a parametrization of the form

$$f : U \times \mathbb{R} \to \mathbb{R}^3, \quad (u,v) \mapsto f(u,v) = \alpha(u) + v\beta(u),$$

where:

- $\alpha : U \to \mathbb{R}^3$ *is a* $\mathcal{C}^1$ *curve called the* ***directrix*** *of* S.
- $\beta : U \to \mathbb{R}^3$ *is a family of non-zero vectors, i.e.,* $\beta(u) \neq 0$ *for all* $u \in U$.

For each $u \in U$*, the* ***generator*** *(or ruling) of* S *is the straight line*

$$\Delta(u) = \{f(u,v) = \alpha(u) + v\beta(u) \mid v \in \mathbb{R}\},$$

which passes through the point $\alpha(u)$ *and has direction* $\beta(u)$.
Thus, the surface S *can be written as the union*

$$S = \bigcup_{u \in U} \Delta(u).$$

In summary, a ruled surface is the union of its generating lines along the directrix curve.

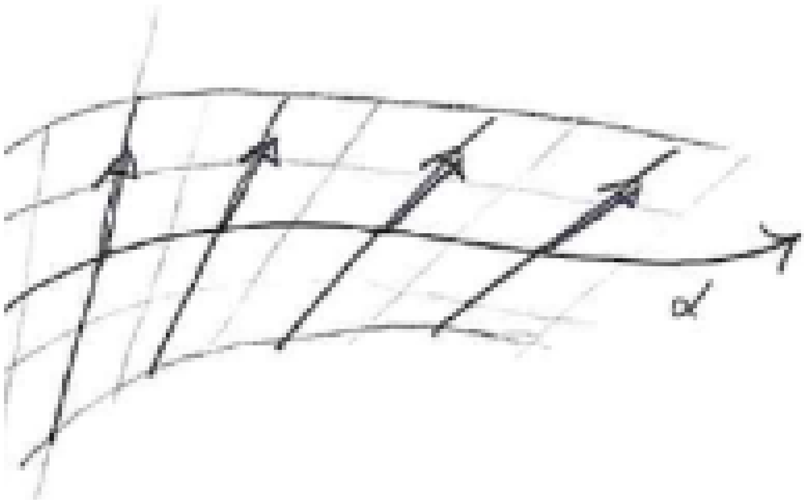


Figure 2.10

Example 2.0.7.

- *Planes*
  Cones
  Cylinders.

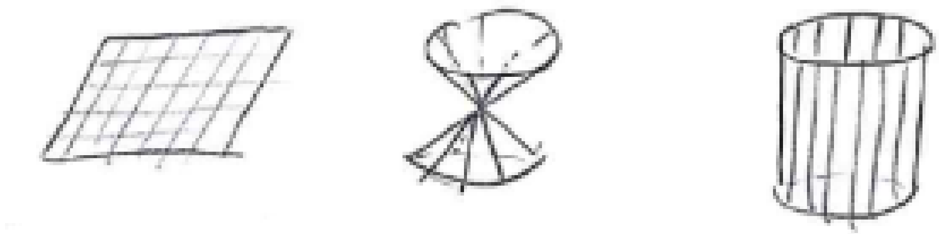


Figure 2.11

- *The one-sheet hyperboloid* $x^2 + y^2 - z^2 = 1$.
  If we call S *the hyperboloid, its intersection with the plane* $\pi = \{x = 1\}$ *gives two lines:*

$$S \cap \pi = \{(1, y, z) \mid y^2 - z^2 = 0\} = \Delta^+ \cup \Delta^-,$$

where

$$\Delta^+ = \{(1, y, z) \mid z = y\}, \quad \Delta^- = \{(1, y, z) \mid z = -y\}$$

are the lines with direction respectively $\beta^+ = (0, 1, 1)$ *and* $\beta^- = (0, 1, -1)$.
Since S *is a surface of revolution, we have:*

$$S = \bigcup_{\varphi \in [0,2\pi[} R_{\varphi}^{\vec{z}}(\Delta^+) \cup R_{\varphi}^{\vec{z}}(\Delta^-)$$

But one line is sufficient to fill S, *because for any* $P \in R_{\vec{z}}^{\varphi}(\Delta^-)$ *there exists a* $\psi \in [0, 2\pi]$ *such that* $P \in R_{\psi}^{\vec{z}}(\Delta^+)$.
In conclusion, we have:

$$\Delta^+ = \{(1, 0, 0) + v(0, 1, 1) \mid v \in \mathbb{R}\};$$

thus,

$$S = \bigcup_{\varphi \in [0,2\pi[} R_{\varphi}^{\vec{z}}(\Delta_+) = \{R_{\varphi}^{\vec{z}}(1, 0, 0) + vR_{\varphi}^{\vec{z}}(0, 1, 1) \mid \varphi \in [0, 2\pi[, v \in \mathbb{R}\},$$

$$= \{f(\varphi, v) = (\cos\varphi, \sin\varphi, 0) + v(-\sin\varphi, \cos\varphi, 1) \mid \varphi \in [0, 2\pi[, v \in \mathbb{R}\}.$$

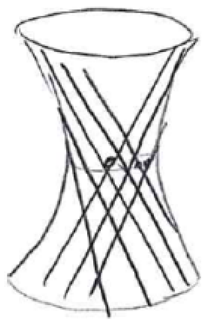


Figure 2.12

The hyperbolic paraboloid $x^2 - y^2 - z = 0$.
If we call it S, its intersection with the planes

$$\pi_u^{\pm} = \{(x, y, z) \mid x \pm y = u\}$$

gives two lines:

$$S \cap \pi_u^+ = \{(x, y, z) \mid x + y = u,\ z = -2uy + u^2\} = \Delta_u^+$$

$$S \cap \pi_u^- = \{(x, y, z) \mid x - y = u,\ z = 2uy + u^2\} = \Delta_u^-$$

And we easily show that:

$$S = \bigcup_{u \in \mathbb{R}} \Delta_u^+ = \bigcup_{u \in \mathbb{R}} \Delta_u^-.$$

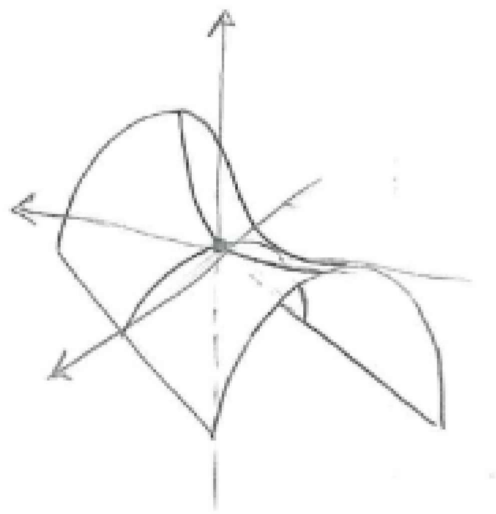


Figure 2.13

2.0.6 Orientable surfaces

Definition 2.0.9.
*Let S be a surface parametrized by $f : U \to \mathbb{R}^3$, and let $p_0 = f(u_0, v_0)$ be a regular point of S. We define the **unit normal vector** to S at p_0 as:*

$$N_S(u_0, v_0) = \frac{\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)}{\|\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)\|}$$

Let $f^* : W \to \mathbb{R}^3$ be another parametric representation of a regular surface S, with parameters (θ, φ), and assume $p_0 = f^*(\theta_0, \varphi_0)$. Then we have:

$$\begin{aligned}
\partial_\theta f^*(\theta_0, \varphi_0) \wedge \partial_\varphi f^*(\theta_0, \varphi_0) &= (\partial_u f(u_0, v_0)\partial_\theta u + \partial_v f(u_0, v_0)\partial_\theta v) \\
&\quad \wedge(\partial_u f(u_0, v_0)\partial_\varphi u + \partial_v f(u_0, v_0)\partial_\varphi v) \\
&= (\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)) (\partial_\theta u \partial_\varphi v - \partial_\varphi u \partial_\theta v) \\
&= (\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)) \cdot \tfrac{\partial(u,v)}{\partial(\theta,\varphi)}
\end{aligned}$$

Thus:

$$\begin{aligned}
N^* &= \frac{\partial_\theta f^*(\theta_0, \varphi_0) \wedge \partial_\varphi f^*(\theta_0, \varphi_0)}{\|\partial_\theta f^*(\theta_0, \varphi_0) \wedge \partial_\varphi f^*(\theta_0, \varphi_0)\|} \\
&= \frac{\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)}{\|\partial_u f(u_0, v_0) \wedge \partial_v f(u_0, v_0)\|} \cdot \frac{\frac{\partial(u,v)}{\partial(\theta,\varphi)}}{\left|\frac{\partial(u,v)}{\partial(\theta,\varphi)}\right|} \\
&= \operatorname{sign}\left(\frac{\partial(u, v)}{\partial(\theta, \varphi)}\right) \cdot N
\end{aligned}$$

Hence, N and N^* have the same orientation if and only if $\frac{\partial(u,v)}{\partial(\theta,\varphi)} > 0$ at point p_0. This shows that N and N^*, the unit normal vectors of two overlapping parametrizations at p_0, differ only by a sign $\pm$.

Definition 2.0.10.
Let S be a regular parametrized surface. The choice of a unit normal vector field

$$N : U \to \mathbb{R}^3$$

*is called an **ORIENTATION** of S.*

- *There are only two possible choices:*

$$N_s = \frac{\partial_u f \wedge \partial_v f}{\|\partial_u f \wedge \partial_v f\|} \quad or \quad N_s = -\frac{\partial_u f \wedge \partial_v f}{\|\partial_u f \wedge \partial_v f\|}$$

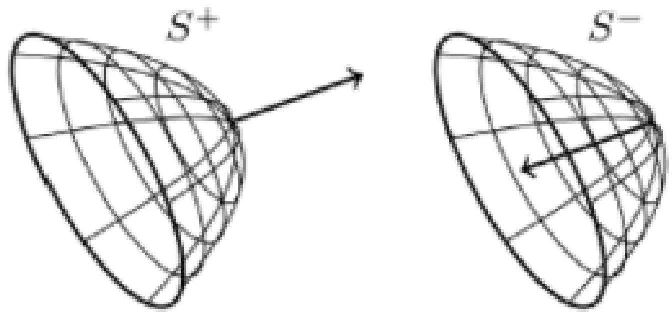


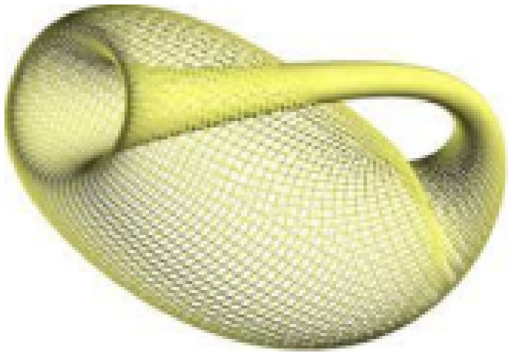
Figure 2.14

Definition 2.0.11.
A surface S is said to be orientable if for any two parametric representations $f(u, v)$ and $g(p, q)$ of S that overlap on a region containing the point $p_0 \in S$, then at every point of this region we have

$$\frac{\partial(u, v)}{\partial(p, q)} > 0.$$

On an orientable surface, the unit normal vector can vary continuously without changing direction. The most standard example of a non-orientable surface is the Möbius strip.
The direction of N is called the outward direction to the surface.

Figure 2.15



Klein bottle

2.1 First Fundamental Form

Just as a curve in $\mathbb{R}^3$ intrinsically depends on two quantities: curvature and torsion, similarly, each surface in $\mathbb{R}^3$ is uniquely determined by two quantities that are locally invariant, called the first and second fundamental forms.

The **first fundamental form** is the usual scalar product in $\mathbb{R}^3$ restricted to the tangent space $T_pS \subset \mathbb{R}^3$. This structure allows us to make measurements on the surface and its tangent spaces.

Let $f = f(u, v)$ be the local coordinates of a surface S of class ≥ 1. The differential of f, denoted by df, is a bijective application from the vector (du, dv) in the uv-plane to the vector

$$df = f_u du + f_v dv,$$

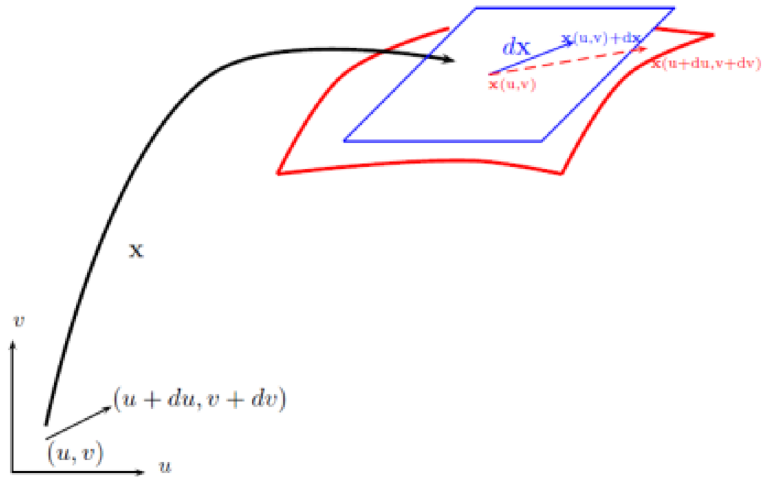
in the tangent plane. In the following figure, we note that the image of the vector (du, dv) is not the vector df in the tangent plane itself, but rather a vector connecting the point $x(u, v)$ on S to the point $f(u + du, v + dv)$, which also lies on the surface.

Let us calculate the quantity:

$$\begin{aligned} I &= df \cdot df = (f_u du + f_v dv) \cdot (f_u du + f_v dv) \\ &= (f_u \cdot f_u) du^2 + 2(f_u \cdot f_v) du\, dv + (f_v \cdot f_v) dv^2 \\ &= E\, du^2 + 2F\, du\, dv + G\, dv^2, \end{aligned}$$

with:

$$E = f_u \cdot f_u, \quad F = f_u \cdot f_v, \quad G = f_v \cdot f_v.$$



Definition 2.1.1.
The quadratic form

$$I(du, dv) = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

is called the first fundamental form of the surface $f(u, v)$, and the coefficients E, F, and G are called the coefficients of the first fundamental form.

Remark 2.1.1.
The form $I(du, dv)$ does not depend on the chosen parametric representation. Indeed, if we consider two parametric representations $f(u, v)$ and $f^*(p, q)$, at every point where

$$f(u, v) = f^*(p(u, v), q(u, v)),$$

then

$$\begin{aligned}
I^*(dp, dq) &= |df^*|^2 = |f_p^* dp + f_q^* dq|^2 \\
&= |f_p^*(p_u du + p_v dv) + f_q^*(q_u du + q_v dv)|^2 \\
&= |(f_p^* p_u + f_q^* q_u)du + (f_p^* p_v + f_q^* q_v)dv|^2 \\
&= |f_u du + f_v dv|^2 \\
&= |df|^2 = I(du, dv).
\end{aligned}$$

Remark 2.1.2.
However, the coefficients of the first fundamental form are not invariant under this transformation, i.e.

$$E(u,v), F(u,v), G(u,v) \neq E^*(p,q), F^*(p,q), G^*(p,q).$$

Indeed:

$$\begin{aligned} E(u,v) &= |f_u|^2 = (f_p^* p_u + f_q^* q_u)\cdot(f_p^* p_u + f_q^* q_u) \\ &= |f_p^*|^2 p_u^2 + 2(f_p^* \cdot f_q^*) p_u q_u + |f_q^*|^2 q_u^2 \\ &= E^*(p,q) p_u^2 + 2F^*(p,q) p_u q_u + G^*(p,q) q_u^2, \end{aligned}$$

$$\begin{aligned} F(u,v) &= f_u \cdot f_v = (f_p^* p_u + f_q^* q_u)\cdot(f_p^* p_v + f_q^* q_v) \\ &= E^*(p,q) p_u p_v + F^*(p,q)(p_u q_v + p_v q_u) + G^*(p,q) q_u q_v, \end{aligned}$$

and

$$\begin{aligned} G(u,v) &= |f_v|^2 = (f_p^* p_v + f_q^* q_v)\cdot(f_p^* p_v + f_q^* q_v) \\ &= E^*(p,q) p_v^2 + 2F^*(p,q) p_v q_v + G^*(p,q) q_v^2. \end{aligned}$$

Theorem 2.1.1. *The quadratic form $I(du,dv)$ is a positive definite form.*

Proof. We can represent the form $I(du,dv)$ by a 2×2 matrix:

$$\begin{aligned} I(du,dv) &= (du,dv)\begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} \\ &= (du,dv)\begin{pmatrix} Edu + Fdv \\ Fdu + Gdv \end{pmatrix} \\ &= Edu^2 + 2Fdudv + Gdv^2. \end{aligned}$$

Then, the quadratic form $I(du,dv)$ is defined by the symmetric matrix

$$A = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}.$$

For it to be positive definite, it is necessary and sufficient that:

$$E > 0 \quad \text{and} \quad EG - F^2 > 0.$$

We have $E = \|f_u\|^2 > 0$ and

$$EF - G^2 = (f_u \cdot f_u)(f_v \cdot f_v) - (f_u \cdot f_v)^2 = \|f_u \wedge f_v\|^2.$$

We will use the formula:

$$(a \wedge b)\cdot(c \wedge d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c).$$

□

There are three quantities that can be computed using the coefficients E, F, and G of the first fundamental form: the length of a curve on the surface, the angle between f_u and f_v, and the area of the surface.

Length of a curve on a surface

Let $f = f(u, v)$ and $\gamma(t) = f(u(t), v(t))$ be a curve on the surface defined by f. We want to calculate:

$$\begin{aligned}
\left|\overbrace{\gamma(a)\gamma(b)}\right| = \int_a^b |\gamma'(t)| dt &= \int_a^b \left|\frac{df}{dt}(u(t), v(t))\right| dt \\
&= \int_a^b \left\| f_u \frac{du}{dt} + f_v \frac{dv}{dt} \right\| dt \\
&= \int_a^b \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{du}{dv} + G\left(\frac{dv}{dt}\right)^2}\, dt \\
&= \int_a^b \sqrt{I\left(\frac{du}{dt}, \frac{du}{dv}\right)}\, dt.
\end{aligned}$$

Example 2.1.1.
Consider the unit sphere

$$f(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

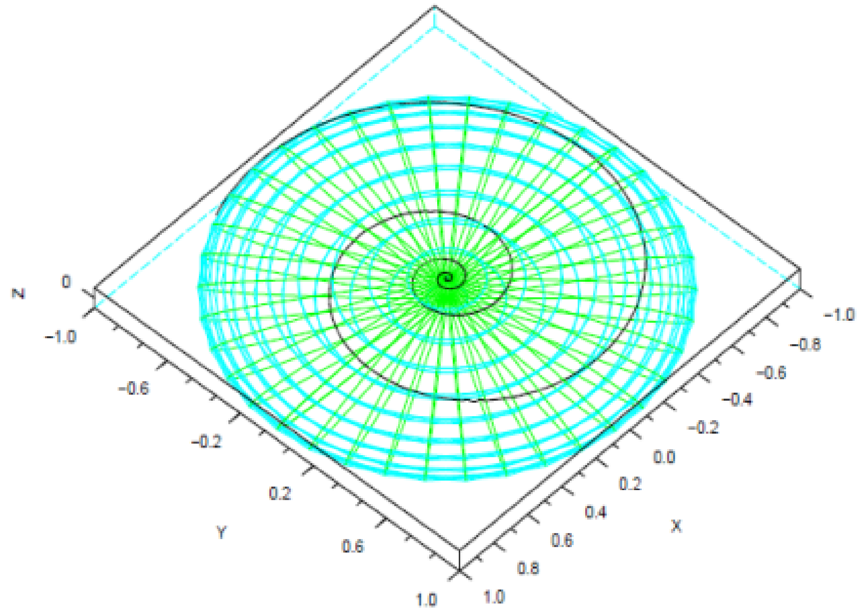
and the curve $\gamma(t) = g(\theta(t), \varphi(t))$ *where*

$$\theta(t) = \ln\left(\cot\left(\frac{\pi}{4} - \frac{t}{2}\right)\right), \quad \varphi(t) = \frac{\pi}{2} - t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

This curve starts from the equator and spirals up to the north pole.

Let us compute:

$$\begin{aligned}
f_u &= (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0), \\
f_v &= (\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi).
\end{aligned}$$



Thus,

$$E = \sin^2 \varphi, \quad F = 0, \quad G = 1,$$

and

$$\frac{d\theta}{dt} = \frac{1}{2\sin\left(\frac{\pi}{4} - \frac{t}{2}\right)\cos\left(\frac{\pi}{4} - \frac{t}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2} - t\right)},$$

$$\frac{d\varphi}{dt} = -1.$$

Which gives

$$I = E\left(\frac{d\theta}{dt}\right)^2 + 2F\frac{d\theta}{dt}\frac{d\varphi}{dt} + G\left(\frac{d\varphi}{dt}\right)^2 = \sin^2 \varphi \frac{1}{\sin^2(\frac{\pi}{2} - t)} + 1 = 2,$$

and finally the length of this curve is

$$\int_0^{\pi/2} \sqrt{I\left(\frac{d\theta}{dt}, \frac{d\varphi}{dt}\right)}\, dt = \int_0^{\pi/2} \sqrt{2}\, dt = \frac{\pi}{\sqrt{2}}.$$

Angle between f_u and f_v

The angle between f_u and f_v:

$$\cos(f_u, f_v) = \frac{f_u \cdot f_v}{|f_u||f_v|} = \frac{F}{\sqrt{EG}}.$$

Corollary 2.1.1.
The vectors f_u and f_v are orthogonal if and only if $F = 0$.

Example 2.1.2.
Let us go back to Example We have

$$E = \sin^2 \varphi, \quad F = 0, \quad G = 1$$

Therefore $\cos(f_u, f_v) = 0$, so f_u and f_v are orthogonal.

Surface Area

Let $\Delta s = du\,dv$ (with $du > 0, dv > 0$) be a small surface element in the uv-plane at the point (u, v). Let ΔS be the image of Δs under f:

$$\Delta S = \|f_u\,du \wedge f_v\,dv\| = \|f_u \wedge f_v\|\,du\,dv = \sqrt{EG - F^2}\,du\,dv.$$

If W^* is the image of a region W whose area we want to compute, then the area is given by:

$$A = \iint_W \sqrt{EG - F^2}\,du\,dv.$$

Example 2.1.3.
Let us take the previous example , with $(\theta, \phi) \in (0, 2\pi) \times (0, \pi)$. We have:

$$E = \sin^2\phi, \quad F = 0, \quad G = 1.$$

The area of this surface is:

$$A_S = \int_0^{\pi} \int_0^{2\pi} \sqrt{EG - F^2}\,d\theta\,d\phi = \int_0^{\pi} \int_0^{2\pi} \sin\phi\,d\theta\,d\phi = 2\pi \int_0^{\pi} \sin\phi\,d\phi = 4\pi.$$

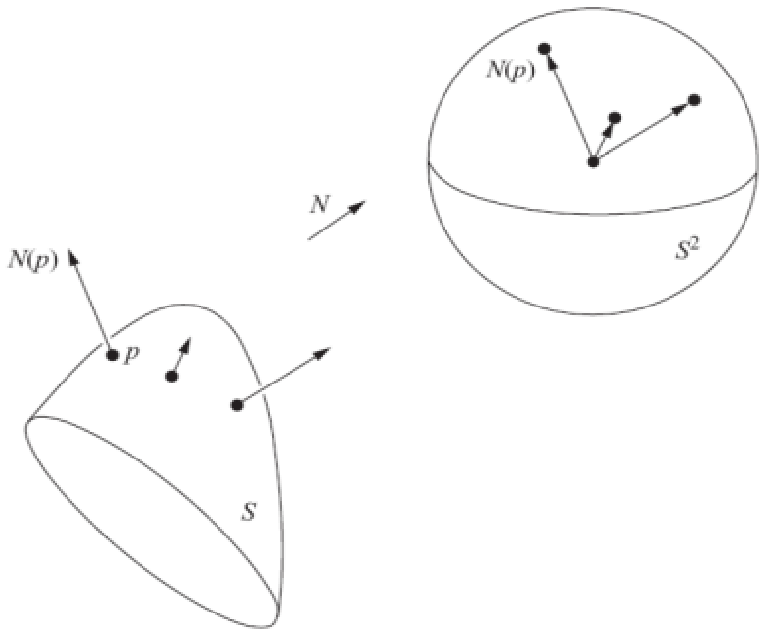
Remark 2.1.3.
If R is the radius of the sphere, then the surface area is given by

$$A_S = 4\pi R^2.$$

2.2 Second fundamental form

2.2.1 Gauss map

Definition 2.2.1.
*Let S be a regular surface with an orientation $\vec{N}$, and let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. The map defined by $\vec{N} : S \to \mathbb{S}^2$ is called the **Gauss map**.*



Gauss map

Remark 2.2.1.

- *Since the surface is regular, the Gauss map at each point $p \in S$ is differentiable, and its differential, denoted dN_p, is a linear map from the tangent plane T_pS to $T_{N(p)}\mathbb{S}^2$. These two planes are parallel, so we consider dN_p as a linear map from T_pS into itself.*
- *When the surface S is a plane, it coincides with its tangent plane at every point, so the Gauss map is constant.*
- *When the surface is a cylinder, the tangent plane is constant along each ruling line of the cylinder, hence the Gauss map sends the surface (cylinder) onto an equator of the unit sphere.*
- *When $S = \mathbb{S}^2$ is the unit sphere, the Gauss map is the identity.*

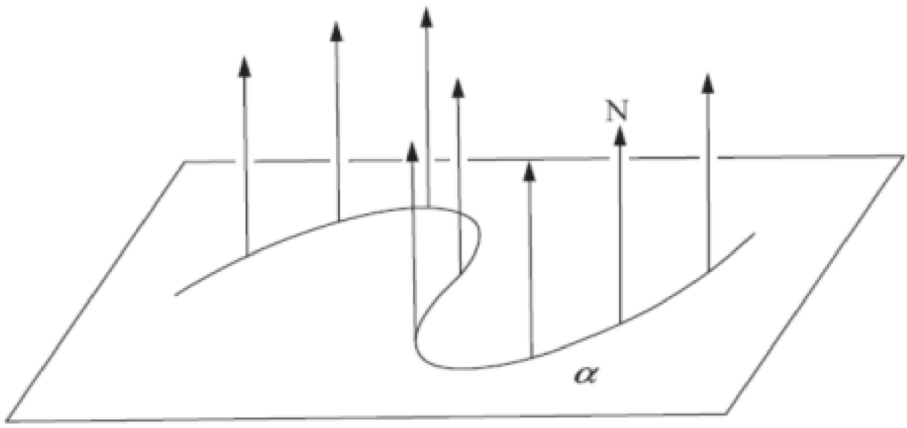
Example 2.2.1.
When the surface S is a plane P given by the equation

$$ax + by + cz + d = 0,$$

it coincides with its tangent plane at every point. The normal vector

$$N = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$$

is constant, and therefore $dN_p = 0$. Hence, the Gauss map is constant.



Gauss map of the plane

2.2.2 Second fundamental form

Let $f(u,v)$ be a parametric representation of a surface of class at least 2. Then, at every point (u,v) of this surface, we can associate a unit normal vector

$$N = \frac{f_u \wedge f_v}{\|f_u \wedge f_v\|}$$

which is a function of (u,v) of class at least C^1.

We denote by $dN = N_u du + N_v dv$ the differential of N. Since

$$0 = d(1) = d(N \cdot N) = 2dN \cdot N,$$

then $N \perp dN$ at the point $f(u,v)$, and at this point, dN lies in the tangent plane. We now consider the quantity:

$$\begin{aligned} II(du,dv) = -df \cdot dN &= -(f_u du + f_v dv) \cdot (N_u du + N_v dv) \\ &= -f_u \cdot N_u\, du^2 - (f_u \cdot N_v + f_v \cdot N_u)\, du\, dv - f_v \cdot N_v\, dv^2 - f_v dv^2 \\ &= L du^2 + M du dv + N dv^2 \end{aligned}$$

with

$$L = -f_u \cdot N_u, \quad M = -(f_u \cdot N_v + f_v \cdot N_u), \quad N = -f_v \cdot N_v.$$

Definition 2.2.2.

- *The quantity $II(du, dv)$ is called the* ***second fundamental form*** *associated with $f(u, v)$.*
- *II is a quadratic form with coefficients L, M, and N, called the* ***coefficients of the second fundamental form****.*

Remark 2.2.2.
II is independent of the parametric representations, and one can show that if $g(p, q)$ is another parametric representation such that

$$f(u, v) = g(p(u, v), q(u, v)),$$

then

$$LN - M^2 = \left(\frac{\partial(p, q)}{\partial(u, v)}\right)^2 (L^* N^* - (M^*)^2).$$

Since f_u and f_v are orthogonal to N at every point (u, v), we have

$$0 = (f_u \cdot N)_u = f_{uu} \cdot N + f_u \cdot N_u,$$

$$0 = (f_u \cdot N)_v = f_{uv} \cdot N + f_u \cdot N_v,$$
$$0 = (f_v \cdot N)_u = f_{vu} \cdot N + f_v \cdot N_u,$$
$$0 = (f_v \cdot N)_v = f_{vv} \cdot N + f_v \cdot N_v,$$

which gives

$$L = f_{uu} \cdot N, \quad M = f_{uv} \cdot N, \quad N = f_{vv} \cdot N,$$

and consequently,

$$II(du, dv) = Ldu^2 + 2Mdudv + Ndv^2 = d^2 f \cdot N,$$

where

$$d^2 f = f_{uu} du^2 + 2f_{uv} dudv + f_{vv} dv^2.$$

Consider a point P on a surface of class $\mathcal{C} \geq 2$, and Q a nearby point on the same surface. Let $f(u, v)$ be a chart containing P and Q. Let

$$\vec{d} = \vec{PQ} \cdot N$$

be the projection of the vector $\overrightarrow{PQ}$ onto the normal vector N. It is clear that d is positive if Q lies above the tangent plane and negative if below.

Suppose $P = f(u, v)$ and $Q = f(u + du, v + dv)$. Then by Taylor-Young formula,

$$f(u+du, v+dv) = f(u, v) + (f_u, f_v)\begin{pmatrix} du \\ dv \end{pmatrix} + \frac{1}{2}\begin{pmatrix} du & dv \end{pmatrix}\begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} + o(du^2 + dv^2).$$

This can be written as

$$f(u + du, v + dv) - f(u, v) = df + \frac{1}{2} d^2 f + o(du^2 + dv^2).$$

Thus,

$$d = \overrightarrow{PQ} \cdot N = \big(f(u + du, v + dv) - f(u, v)\big) \cdot N = \left(df + \frac{1}{2} d^2 f + o(du^2 + dv^2)\right) \cdot N.$$

Since $df \cdot N = 0$, we get

$$d = \frac{1}{2}d^2 f \cdot N + o(du^2 + dv^2) = \frac{1}{2}II + o(du^2 + dv^2) = \delta + o(du^2 + dv^2),$$

where the function

$$\delta = \frac{1}{2}II(du, dv) = \frac{1}{2}(Ldu^2 + 2Mdudv + Ndv^2)$$

is called the ***osculating paraboloid*** *at the point P, and the nature of this paraboloid depends on the sign of $LN - M^2$.*

We distinguish four cases:

1. ***Elliptic case*** $(LN - M^2 > 0)$
   A point is called elliptic if $LN - M^2 > 0$. At this point, the function δ maintains a constant sign in a neighborhood of P, and the set of such points lies on one side of the tangent plane at P.

2. ***Hyperbolic case*** $(LN - M^2 < 0)$
   A point is hyperbolic if $LN - M^2 < 0$. At this point, there exist two lines dividing the tangent plane into four distinct parts, on each of which $d = \overrightarrow{PQ} \cdot N$ has a constant sign.

3. ***Parabolic case*** $(LN - M^2 = 0$ *and* $L^2 + M^2 + N^2 \neq 0)$
   A point on the surface is parabolic if $LN - M^2 = 0$ without all coefficients L, M, N being zero. On the tangent plane at P, there exists a planar curve defined by $\delta = 0$ such that on this curve $\delta = 0$, and outside the curve the sign of $d = \overrightarrow{PQ} \cdot N$ is constant.

4. ***Planar case*** $(LN - M^2 = 0$ *and* $L^2 + M^2 + N^2 = 0)$
   In this case, if $L = M = N = 0$, the quadratic form II is identically zero. To define d, we must go to order 3 and write

   $$d = \frac{1}{2}II + \frac{1}{6}d^3 f \cdot N + o\big((du^2 + dv^2)^{3/2}\big).$$

   Then,

   $$\delta = \frac{1}{6}d^3 f \cdot N,$$

   which is a cubic form, and consequently there exist three lines along which δ is zero.

Example 2.2.2.
An elliptical paraboloid is the shape of a bowl or an inverted bowl, given by

$$f(u,v) = (u, v, u^2 + v^2)$$

or

$$f(u,v) = (u, v, -(u^2 + v^2)).$$

In the first case, we have

$$f_u = (1, 0, 2u), \quad f_v = (0, 1, 2v), \quad \text{and} \quad N = (0, 0, 1),$$

so

$$L = f_{uu} \cdot N = 2, \quad N = f_{vv} \cdot N = 2, \quad \text{and} \quad M = f_{uv} \cdot N = 0.$$

In the second case, we have

$$f_u = (1, 0, -2u), \quad f_v = (0, 1, -2v), \quad \text{and} \quad N = (0, 0, 1),$$

so

$$L = f_{uu} \cdot N = -2, \quad N = f_{vv} \cdot N = -2, \quad \text{and} \quad M = f_{uv} \cdot N = 0.$$

In any case, we get

$$LN - M^2 = 4,$$

which corresponds to the elliptic case.

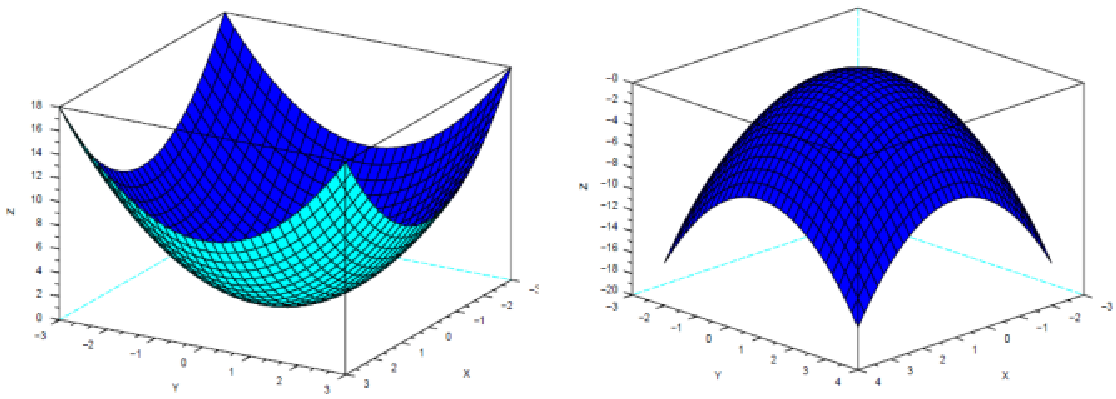


Figure 2.23

Example 2.2.3.
A hyperbolic paraboloid is the shape of a saddle. We have

$$f(u,v) = (u, v, u^2 - v^2).$$

We get

$$f_u = (1, 0, 2u), \quad f_v = (0, 1, -2v), \quad \textit{and} \quad N = (0, 0, 1),$$

so

$$L = f_{uu} \cdot N = 2, \quad N = f_{vv} \cdot N = -2, \quad \textit{and} \quad M = f_{uv} \cdot N = 0.$$

Thus,

$$LN - M^2 = -4.$$

We note that

$$\delta = \frac{1}{2}\left(Ldu^2 + 2Mdudv + Ndv^2\right) = du^2 - dv^2,$$

and on the two lines $u = v$ and $u = -v$, we have $\delta = 0$ (this corresponds to the hyperbolic case).

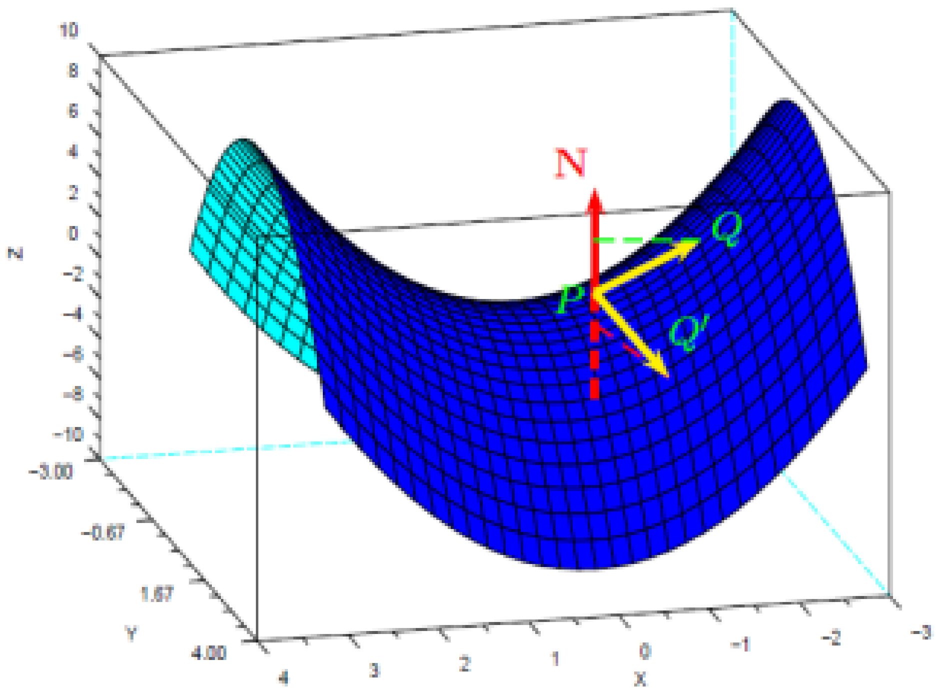


Figure 2.24

Example 2.2.4.
The torus is given by

$$f(\theta,\varphi)=\big((b+a\sin\theta)\cos\varphi,\ (b+a\sin\theta)\sin\varphi,\ a\cos\theta\big),$$

and the normal vector is

$$N=(-\sin\theta\cos\varphi,\ -\sin\theta\sin\varphi,\ -\cos\theta).$$

Thus,

$$L=a,\quad N=(b+a\sin\theta)\sin\theta,\quad \text{and}\quad M=0.$$

Consequently,

$$LN-M^2=a(b+a\sin\theta)\sin\theta,$$

which is zero for $\theta=0$ or π, giving two circles representing δ.

All points on the outer face

$$\{f(\theta,\varphi)\mid(\theta,\varphi)\in]0,\pi[\times[0,2\pi]\}$$

are elliptic points, and all points on the inner face

$$\{f(\theta,\varphi)\mid(\theta,\varphi)\in]\pi,2\pi[\times[0,2\pi]\}$$

are hyperbolic points.

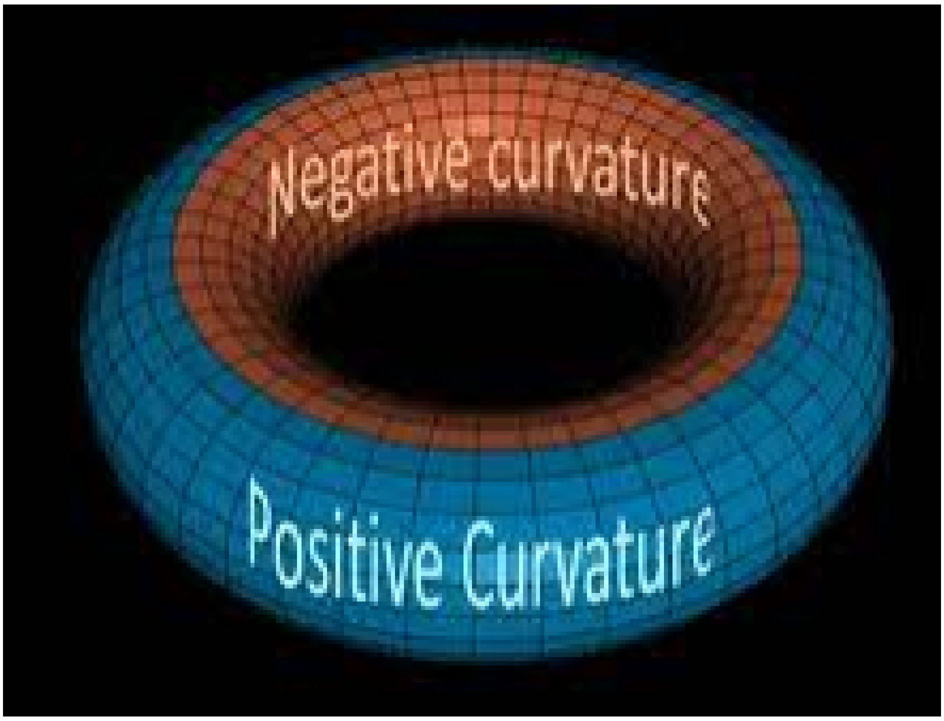


Figure 2.25

Example 2.2.5.
(Monkey Saddle) *For this surface, the parametric equation is:*

$$f(u,v) = \big(u, v, u(u^2 - 3v^2)\big).$$

The normal vector is

$$N = \frac{1}{\sqrt{1 + 9(u^2 + v^2)^2}}\big(3(v^2 - u^2),\ 6uv,\ 1\big).$$

Therefore,

$$L = \frac{6u}{\sqrt{1 + 9(u^2 + v^2)^2}}, \quad N = \frac{-6u}{\sqrt{1 + 9(u^2 + v^2)^2}}, \quad M = \frac{-6v}{\sqrt{1 + 9(u^2 + v^2)^2}}.$$

We notice that all these coefficients vanish at the point $P = (0, 0, 0)$.

2.2.3 Gaussian curvature

Definition 2.2.3.
Let C be a regular curve on a surface S passing through a point $p \in S$. Let k be the curvature of C at p, $\vec{n}$ the normal vector to the curve C at p, and $\vec{N}$ the normal vector to the surface S at p. The real number

$$k_n = k \cdot \langle \vec{n}, \vec{N} \rangle$$

is called the ***normal curvature*** *of the curve $C \subset S$ at point p.*

Remark 2.2.3.
Let θ be the angle between the vectors $\vec{n}$ (the normal to the curve) and $\vec{N}$ (the normal to the surface at point p). Then the normal curvature is given by:

$$k_n = k \cdot \cos(\theta),$$

which is the norm of the projection of the curvature vector $k\vec{n}$ onto the normal direction of the surface at p.

Let C be a regular curve lying on the surface S, parametrized by arc length as $\alpha(s)$, such that $\alpha(0) = p$. Then the tangent vector at p is $\alpha'(0) \in T_p(S)$.

For all s, we have:

$$\langle N \circ \alpha(s), \alpha'(s) \rangle = 0,$$

since the surface normal is perpendicular to the curve tangent. Differentiating this identity with respect to s, we get:

$$\langle (N \circ \alpha)'(s), \alpha'(s) \rangle + \langle N \circ \alpha(s), \alpha''(s) \rangle = 0,$$

so at $s = 0$, we have:

$$\langle N, \alpha''(0) \rangle = -\langle d_pN(\alpha'(0)), \alpha'(0) \rangle.$$

This leads to:

$$II_p(\alpha'(0)) = -\langle d_pN(\alpha'(0)), \alpha'(0) \rangle = \langle N, \alpha''(0) \rangle = k_n(p),$$

which expresses the second fundamental form in terms of the normal curvature.

Proposition 2.2.1.
All curves lying on a surface S that share the same tangent line at a point $p \in S$ have the same normal curvature at that point.

Proof.
Let C and C' be two curves on the surface S that have the same tangent line at a point $p \in S$. Let α and β be arc-length parametrizations of C and C', respectively, such that:

$$\alpha(0) = p = \beta(0).$$

By assumption, the two curves share the same tangent vector at p, so:

$$\alpha'(0) = \pm\beta'(0).$$

Then, the normal curvature of C at p is:

$$k_n(\alpha(0)) = II_p(\alpha'(0)) = -\langle d_pN(\alpha'(0)), \alpha'(0)\rangle.$$

Since $\alpha'(0) = \pm\beta'(0)$, and the second fundamental form is quadratic in the direction, we have:

$$k_n(\alpha(0)) = -(\pm 1)^2\langle d_pN(\beta'(0)), \beta'(0)\rangle = II_p(\beta'(0)) = k_n(\beta(0)).$$

Therefore, the normal curvature depends only on the direction of the tangent vector, and not on the particular curve. This justifies the definition of the normal curvature of S at p in the direction $w \in T_p(S)$.

□ □

Definition 2.2.4.
*Let w be a unit vector in the tangent plane $T_p(S)$. The intersection of the surface S with the plane determined by w and the normal vector $N(p)$ is called the **normal section** of S at p in the direction of w.*

Remark 2.2.4.
The normal section defined in this way is a regular curve whose normal vector at the point p is $\pm N(p)$. Its curvature at p is equal to the absolute value of the normal curvature of the surface at p in the corresponding direction.

Definition 2.2.5.
*The opposites of the eigenvalues of the differential d_pN of the Gauss map are called the **principal curvatures** of the surface S at the point p. The directions given by the associated eigenvectors are called the **principal directions** at p.*

Remark 2.2.5.
The endomorphism d_pN is self-adjoint, and therefore diagonalizable in an orthonormal basis. As a result, there exist two orthogonal vectors (e_1, e_2) in $T_p(S)$ and two real scalars $(-k_1, -k_2) \in \mathbb{R}^2$ such that:

$$d_pN(e_1) = -k_1e_1 \quad \text{and} \quad d_pN(e_2) = -k_2e_2.$$

Therefore, the principal directions are orthogonal.

Definition 2.2.6.
If a connected regular curve $C \subset S$ is such that its tangent vectors at every point are principal directions of the surface, then C is called a ***line of curvature*** *of S.*

Proposition 2.2.2.
Let C be a connected regular curve on a surface S. Then C is a line of curvature if and only if, for every parametrization $\alpha : I \to S$ of C, there exists a function $\lambda : I \to \mathbb{R}$ such that:

$$\forall t \in I, \quad (N \circ \alpha)'(t) = \lambda(t) \cdot \alpha'(t).$$

In that case, for every $t \in I$, the principal curvature of S at the point $\alpha(t)$ is given by $-\lambda(t)$.

Proof.
Direct implication: Assume that C is a line of curvature. Then, for any parametrization $\alpha : I \to S$, the tangent vector $\alpha'(t)$ is, by definition, a principal direction at the point $\alpha(t)$. This means that $\alpha'(t)$ is an eigenvector of the differential $d_{\alpha(t)}N$. Therefore, we can write:

$$d_{\alpha(t)}N(\alpha'(t)) = -k(t)\,\alpha'(t),$$

for some scalar function $k(t)$. Define $\lambda(t) = -k(t)$, and we obtain:

$$(N \circ \alpha)'(t) = \lambda(t)\,\alpha'(t),$$

as required.

Converse implication: Assume that there exists a function $\lambda : I \to \mathbb{R}$ such that:

$$(N \circ \alpha)'(t) = \lambda(t)\,\alpha'(t).$$

Since $(N \circ \alpha)'(t) = d_{\alpha(t)}N(\alpha'(t))$, it follows that $\alpha'(t)$ is an eigenvector of $d_{\alpha(t)}N$, meaning that the tangent vector to the curve at $\alpha(t)$ is a principal direction. Therefore, C is a line of curvature, and $-\lambda(t)$ is the corresponding principal curvature. □ □

Remark 2.2.6.
The principal curvatures allow us to compute the normal curvature in any direction. Let (e_1, e_2) be an orthonormal basis of $T_p(S)$ formed by eigenvectors of d_pN, and such that it preserves the orientation given by the normal vector N. Let $w \in T_p(S)$ be a unit vector, and let θ be the angle between e_1 and w, measured with respect to the orientation of (e_1, e_2).
Then, we can express w as:

$$w = \cos(\theta)\, e_1 + \sin(\theta)\, e_2,$$

and the normal curvature of S at point p in the direction w is:

$$k_n(p) = II_p(w) = -\langle d_pN(w), w\rangle.$$

Using linearity and symmetry of d_pN, we compute:

$$\begin{aligned} k_n(p) &= -\left\langle d_pN\left(\cos\theta\, e_1 + \sin\theta\, e_2\right), \cos\theta\, e_1 + \sin\theta\, e_2\right\rangle \\ &= \langle -k_1 \cos\theta\, e_1 - k_2 \sin\theta\, e_2,\ \cos\theta\, e_1 + \sin\theta\, e_2\rangle \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta. \end{aligned}$$

Thus, this formula gives the expression of the second fundamental form in the orthonormal basis (e_1, e_2).

Example 2.2.6.
Let us show that the horizontal and vertical normal sections are lines of curvature.
Horizontal sections: *A horizontal section at height $a \in \mathbb{R}$ is parametrized by:*

$$\alpha(t) = (\cos t, \sin t, a), \quad t \in (-4\pi, 4\pi).$$

The Gauss map composed with α is:

$$(N \circ \alpha)(t) = (\cos t, \sin t, 0),$$

and its derivative is:

$$(N \circ \alpha)'(t) = (-\sin t, \cos t, 0) = \alpha'(t).$$

Therefore, by Proposition, this curve is a line of curvature.
Vertical sections: *A vertical section at angle $\theta \in (0, 4\pi)$ is parametrized by:*

$$\beta(t) = (\cos\theta, \sin\theta, t), \quad t \in \mathbb{R}.$$

Then:

$$(N \circ \beta)(t) = (\cos\theta, \sin\theta, 0),$$

so:

$$(N \circ \beta)'(t) = (0, 0, 0) = 0 \cdot \beta'(t).$$

Hence, again by Proposition, this curve is a line of curvature.

Definition 2.2.7.
Let $p \in S$, and consider the differential of the Gauss map $d_pN : T_p(S) \to T_p(S)$. The ***determinant*** *of this linear map is the* ***Gaussian curvature*** *K_p of the surface at p, and the* ***opposite of half of its trace*** *is the* ***mean curvature*** *H_p at p.*
In terms of the principal curvatures k_1 and k_2, we have:

$$K_p = k_1 \cdot k_2, \qquad H_p = \frac{k_1 + k_2}{2}.$$

Let us express these curvature quantities in the basis $(\partial_u\Phi, \partial_v\Phi)$ of the tangent plane $T_p(S)$.

Let $I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ be the matrix of the first fundamental form, and $II_p = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ the matrix of the second fundamental form, and let M_p denote the matrix of the differential d_pN of the Gauss map in this same basis.

Then, for all $w_1, w_2 \in T_p(S)$, we have:

$$w_2^\top II_p w_1 = \langle -d_pN(w_1), w_2 \rangle = -\langle M_p w_1, w_2 \rangle = -w_2^\top I_p M_p w_1.$$

From this, we deduce that:

$$II_p = -I_p M_p \quad \Rightarrow \quad M_p = -I_p^{-1} II_p.$$

Hence, the matrix M_p is given by:

$$M_p = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

From this, the Gaussian and mean curvatures are computed as:

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{eG + gE - 2fF}{2(EG - F^2)}.$$

Example 2.2.7.
Let us compute the Gaussian curvature, mean curvature, and principal curvatures at a point p on a cylinder and on a plane.

Cylinder: *At any point on the cylinder, we have the following coefficients:*

$$E = 1, \quad F = 0, \quad G = 1, \quad e = 1, \quad f = 0, \quad g = 0.$$

Then the curvatures are given by:

$$K = \frac{eg - f^2}{EG - F^2} = \frac{0 - 0}{1 \cdot 1 - 0^2} = 0, \qquad H = \frac{eG + gE - 2fF}{2(EG - F^2)} = \frac{1 \cdot 1 + 0 \cdot 1 - 0}{2(1 \cdot 1 - 0)} = \frac{1}{2}.$$

Using the root formulas:

$$k_1 = 0, \quad k_2 = 1.$$

Plane: *At any point on the plane, the coefficients are:*

$$E = 1, \quad F = 0, \quad G = 1, \quad e = 0, \quad f = 0, \quad g = 0.$$

Then:

$$K = \frac{0 \cdot 0 - 0^2}{1 \cdot 1 - 0} = 0, \qquad H = \frac{0 + 0 - 0}{2} = 0.$$

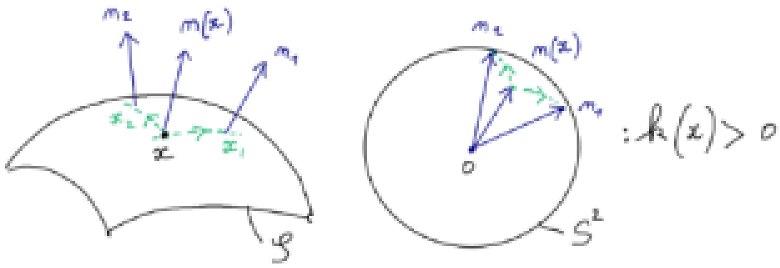
Thus, the principal curvatures are:

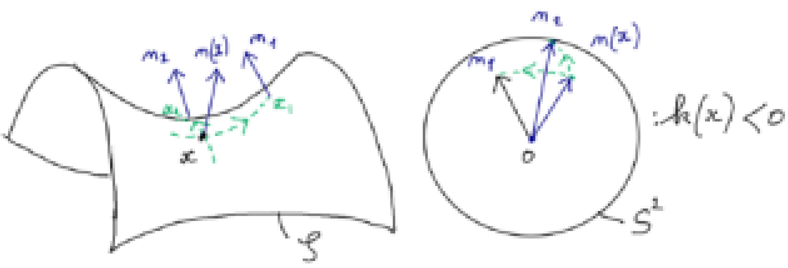
$$k_1 = 0, \quad k_2 = 0.$$

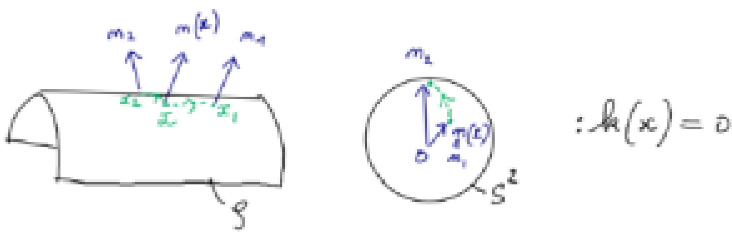
Definition 2.2.8.
Let p be a point on a regular surface S. We classify the point p as follows:

- *p is **elliptic** if $K_p > 0$,*
- *p is **hyperbolic** if $K_p < 0$,*
- *p is **parabolic** if $K_p = 0$ but $d_pN \neq 0$,*
- *p is **planar** if $d_pN = 0$.*

*Moreover, a point p is called an **umbilic point** if the principal curvatures are equal: $k_1 = k_2$.*







Remark 2.2.7. *The Gaussian curvature of a plane is zero because both principal curvatures are zero. When you bend a pizza slice in one direction, one principal curvature becomes nonzero while the other remains zero, keeping the Gaussian curvature zero. This prevents the pizza from bending in the other direction, keeping it rigid and preventing toppings from falling off. Thus, folding the slice sideways stabilizes it, illustrating Gauss's remarkable theorem.*

Remark 2.2.8.
The type of a point determines the local position of the surface relative to its tangent plane at that point.
For example, on a cylinder, all points are ***hyperbolic****.*

Proposition 2.2.3.
If all points of a connected surface S are umbilic, then the surface lies entirely in a plane or on a sphere.

Proof.
Local proof: Let $\Phi : U \to S$ be a local parametrization of the surface with image $V \subset S$, where V is connected. Let $p \in V$. Since p is an umbilic point, the differential of the Gauss map at p satisfies:

$$d_pN = k_1 \cdot \mathrm{id} = \lambda(p) \cdot \mathrm{id}.$$

We aim to show that λ is constant on V.

Let $w = a\,\partial_u\Phi + b\,\partial_v\Phi \in T_p(S)$. Then:

$$d_pN(w) = a\,\partial_u N + b\,\partial_v N = \lambda(p)(a\,\partial_u\Phi + b\,\partial_v\Phi).$$

Since this holds for all such vectors w, we conclude:

$$\partial_u N = \lambda\,\partial_u\Phi, \qquad \partial_v N = \lambda\,\partial_v\Phi.$$

Differentiating the first equation with respect to v and the second with respect to u, we obtain:

$$\partial_v\lambda\,\partial_u\Phi + \lambda\,\partial_{uv}\Phi = \partial_{uv}N, \partial_u\lambda\,\partial_v\Phi + \lambda\,\partial_{uv}\Phi = \partial_{uv}N.$$

Subtracting both equations yields:

$$\partial_v\lambda\,\partial_u\Phi - \partial_u\lambda\,\partial_v\Phi = 0.$$

Since $(\partial_u\Phi, \partial_v\Phi)$ form a basis of the tangent space, this implies:

$$\partial_u\lambda = \partial_v\lambda = 0,$$

so λ is constant on V.

Now, consider two cases:

Case 1: $\lambda = 0$ Then $\partial_u N = \partial_v N = 0$, so N is constant on V. Since N is the normal vector to the surface, this implies the surface lies entirely in a plane orthogonal to N. Furthermore,

$$\partial_u\langle\Phi, N\rangle = \langle\partial_u\Phi, N\rangle = 0, \quad \partial_v\langle\Phi, N\rangle = \langle\partial_v\Phi, N\rangle = 0,$$

so $\langle\Phi, N\rangle$ is constant. Hence, $\Phi(U) \subset$ plane.

Case 2: $\lambda \neq 0$ We define the function:

$$y = \Phi - \frac{1}{\lambda}N.$$

Differentiating:

$$\partial_u y = \partial_u\Phi - \frac{1}{\lambda}\partial_u N = \partial_u\Phi - \frac{1}{\lambda}\lambda\,\partial_u\Phi = 0,$$
$$\partial_v y = \partial_v\Phi - \frac{1}{\lambda}\partial_v N = \partial_v\Phi - \frac{1}{\lambda}\lambda\,\partial_v\Phi = 0.$$

Hence, y is constant on V, so the surface lies on the sphere centered at y with radius $\frac{1}{|\lambda|}$. Indeed:

$$\|\Phi(u,v) - y\|^2 = \left\|\frac{1}{\lambda}N\right\|^2 = \frac{1}{\lambda^2}.$$

Global conclusion: We have shown that S is locally included in a plane or a sphere. To extend this to the whole surface, let $p \in S$. There exists a neighborhood $V_p \subset S$ around p contained in a plane or sphere.

Let $r \in S$. Since S is connected, there exists a continuous path $\alpha : [0,1] \to S$ such that $\alpha(0) = p$ and $\alpha(1) = r$. For each $\alpha(t)$, there exists a neighborhood $V_t \subset S$ included in a plane or sphere. The family of inverse images $\alpha^{-1}(V_t)$ forms an open cover of the compact interval $[0,1]$, so a finite subcover $V_{t_1}, \ldots, V_{t_n}$ exists.

Since each neighborhood along the path lies in a plane or sphere, and they overlap, the surface must globally lie in the same plane or sphere. Hence, S is included in a plane or a sphere.

□

Chapter 3

Minimal Surfaces

In this section, we introduce the concept of **minimal surfaces** and study classical examples of such surfaces.

3.0.1 Definition and Properties

Definition 3.0.1.
*Let S be a regular (parametrized) surface. We say that S is **minimal** if its **mean curvature** is zero at every point.*

Remark 3.0.1.
Since $H = \frac{k_1+k_2}{2}$, a minimal surface satisfies $k_1 = -k_2$. Let us now justify the term minimal.

Characterization

Definition 3.0.2.
*Let $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Let $D \subset U$ be a bounded domain, and let $h : D \to \mathbb{R}$ be a differentiable function. We define the **normal variation** of $\beta(D)$ determined by h as:*

$$\varphi : D \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$$
$$((u,v),t) \mapsto \beta(u,v) + t \cdot h(u,v) \cdot N(u,v)$$

We study the normal variation for fixed $t \in (-\varepsilon, \varepsilon)$. To do so, we define:

$$\beta_t : D \to \mathbb{R}^3, \quad (u,v) \mapsto \varphi(u,v,t)$$

which is a parametrized surface. We compute the partial derivatives of the variation:

$$\partial_u \beta_t = \partial_u \beta + t \cdot \partial_u h \cdot N + t \cdot h \cdot \partial_u N$$
$$\partial_v \beta_t = \partial_v \beta + t \cdot \partial_v h \cdot N + t \cdot h \cdot \partial_v N$$

Using the facts that $\partial_u \beta \perp N$, $|N| = 1$, and $N \perp \partial_u N, \partial_v N$, we obtain:

$$E_t = E + th\left(\langle \partial_u \beta, \partial_u N\rangle + \langle \partial_u \beta, \partial_u N\rangle\right) + t^2(\partial_u h)^2 + t^2h^2\langle \partial_u N, \partial_u N\rangle$$
$$F_t = F + th\left(\langle \partial_u \beta, \partial_v N\rangle + \langle \partial_v \beta, \partial_u N\rangle\right) + t^2\partial_u h \cdot \partial_v h + t^2h^2\langle \partial_u N, \partial_v N\rangle$$
$$G_t = G + th\left(\langle \partial_v \beta, \partial_v N\rangle + \langle \partial_v \beta, \partial_v N\rangle\right) + t^2(\partial_v h)^2 + t^2h^2\langle \partial_v N, \partial_v N\rangle$$

Moreover, using the relations:

$$\langle \partial_u \beta, \partial_u N\rangle = -e, \quad \langle \partial_u \beta, \partial_v N\rangle + \langle \partial_v \beta, \partial_u N\rangle = -2f, \quad \langle \partial_v \beta, \partial_v N\rangle = -g$$

we get:

$$E_t = E - 2the + t^2(\partial_u h)^2 + t^2h^2\langle \partial_u N, \partial_u N\rangle$$
$$F_t = F - 2thf + t^2\partial_u h \cdot \partial_v h + t^2h^2\langle \partial_u N, \partial_v N\rangle$$
$$G_t = G - 2thg + t^2(\partial_v h)^2 + t^2h^2\langle \partial_v N, \partial_v N\rangle$$

Then:

$$E_tG_t - (F_t)^2 = (EG - F^2) - 2th(Eg + eG - 2fF) + t^2R = (EG - F^2)(1 - 4thH) + t^2R$$

where R is a polynomial in t.

Thus, for ε small enough, for all $t \in (-\varepsilon, \varepsilon)$, β_t is a regular parametrized surface (regular if and only if $\partial_u \beta_t, \partial_v \beta_t$ are linearly independent, i.e., $E_tG_t - F_t^2 \neq 0$).

Let us compute the area $A(t)$ of the surface β_t:

$$A(t) = \int_D \sqrt{E_tG_t - F_t^2}\, du\, dv = \int_D \sqrt{(1 - 4thH + t^2R)(EG - F^2)}\, du\, dv$$

This function is differentiable on $(-\varepsilon, \varepsilon)$, and in particular:

$$A'(0) = -\int_D 2hH\sqrt{EG - F^2}\, du\, dv$$

Remark 3.0.2.

Let us describe geometrically the operations above: performing a normal variation on β amounts to slightly deforming the surface over the domain D. We then compute $A(t)$, the area of the surface after the deformation. The goal is to show that for a minimal surface, the minimum of $A(t)$ is attained at $t = 0$, that is, when we are measuring the area of the original surface.

Proposition 3.0.1. *(Meusnier, 1776)*
Let $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface. Then β is minimal if and only if for every bounded open subset $D \subset U$, and for every normal variation of $\beta(D)$, we have $A'(0) = 0$.

Proof.
Forward direction: If the surface is minimal, then clearly $A'(0) = 0$ for every open set and every normal variation, since $H = 0$ at every point.

Converse: Suppose now there exists a point $q \in U$ such that $H(q) \neq 0$. Then, by the continuity of H, there exists a neighborhood $D_0 \subset U$ of q on which H does not vanish. Let $h = H$ on D_0. Then the normal variation defined by h leads to $A'(0) < 0$, contradicting our assumption.

□ □

Therefore, a minimal surface is a surface such that every open subset is a critical point of the area functional associated with any normal variation. Note that this critical point is not necessarily a minimum. The term "minimal surface" comes from the fact that the first known example of such a surface is the *catenoid*, which minimizes area between two parallel circles.

1.3 Mean Curvature Vector

Definition 3.0.3.
For a (parametrized) regular surface, the mean curvature vector is defined as

$$\vec{H} = H \cdot \vec{N}$$

where H is the scalar mean curvature and $\vec{N}$ is the unit normal vector.

Remark 3.0.3.
The mean curvature vector of a minimal surface is therefore zero. We will now see that this vector allows us to characterize minimal surfaces.

Proposition 3.0.2.

Let $\beta : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrized surface such that

$$\langle \partial_u \beta, \partial_u \beta \rangle = \langle \partial_v \beta, \partial_v \beta \rangle \quad (*) \quad \text{and} \quad \langle \partial_u \beta, \partial_v \beta \rangle = 0 \quad (**)$$

Then,

$$\partial_{uu}\beta + \partial_{vv}\beta = 2\lambda \vec{H}$$

where $\lambda = \langle \partial_u \beta, \partial_u \beta \rangle$.

Proof.
By differentiating and combining conditions (*) and (**), we obtain:

$$\langle \partial_{uu}\beta, \partial_u \beta \rangle = \langle \partial_{uv}\beta, \partial_v \beta \rangle = -\langle \partial_u \beta, \partial_{vv}\beta \rangle$$

Thus,

$$\langle \partial_{uu}\beta + \partial_{vv}\beta, \partial_u \beta \rangle = 0 \quad \text{and similarly} \quad \langle \partial_{uu}\beta + \partial_{vv}\beta, \partial_v \beta \rangle = 0$$

So, $\partial_{uu}\beta + \partial_{vv}\beta$ is orthogonal to the plane spanned by $\partial_u\beta$ and $\partial_v\beta$, hence it is parallel to $\vec{N}$, the normal vector.

We also have the expression for the mean curvature:

$$H = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

Under our assumptions, we obtain:

$$H = \frac{e + g}{2\langle \partial_u\beta, \partial_u\beta \rangle}$$

So:

$$2\lambda\langle \vec{N}, \vec{H} \rangle = 2\lambda H = e + g = \langle \vec{N}, \partial_{uu}\beta + \partial_{vv}\beta \rangle$$

Thus:

$$\langle \vec{N}, 2\lambda\vec{H} - (\partial_{uu}\beta + \partial_{vv}\beta) \rangle = 0$$

But since $\vec{N}$ and $2\lambda\vec{H} - (\partial_{uu}\beta + \partial_{vv}\beta)$ are colinear and $\vec{N} \neq 0$, we conclude:

$$2\lambda\vec{H} = \partial_{uu}\beta + \partial_{vv}\beta$$

This property gives us a new characterization of minimal surfaces.

□ □

Corollary 2:
Let $\beta : (u, v) \in U \mapsto (x(u, v), y(u, v), z(u, v))$ be a parametrized surface satisfying the same conditions as in the previous proposition. Then, β is minimal **if and only if** x, y, and z are harmonic functions.

Proof.
The surface β is minimal if and only if the mean curvature H is zero, which is equivalent to

$$\partial_{uu}\beta + \partial_{vv}\beta = 0.$$

But we have:

$$\partial_{uu}\beta + \partial_{vv}\beta = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

where Δx, Δy, and Δz denote the Laplacians of the coordinate functions.

Hence, the vanishing of $\partial_{uu}\beta + \partial_{vv}\beta$ is equivalent to $\Delta x = \Delta y = \Delta z = 0$, i.e., x, y, and z are harmonic functions. This concludes the proof

□ □

1.4 Isothermal Parametrization

Definition 3.0.4.

A parametrization satisfying the conditions of the previous proposition is called isothermal.

Proposition 3.0.3.
A parametrization of a surface is isothermal if and only if it is conformal.

Proof.
Let $\Phi : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$. Then the differential $d_q\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies:

$$d_q\Phi(a_1, a_2) = \partial_u\Phi(q) \cdot a_1 + \partial_v\Phi(q) \cdot a_2.$$

Thus,

$$\langle d_q\Phi(a_1, a_2), d_q\Phi(b_1, b_2)\rangle = \langle \partial_u\Phi, \partial_u\Phi\rangle a_1b_1 + \langle \partial_v\Phi, \partial_v\Phi\rangle a_2b_2 + \langle \partial_u\Phi, \partial_v\Phi\rangle(a_1b_2 + a_2b_1),$$

and the rest of the proof follows easily.

□ □

lemma 3.0.1. *Every regular surface admits a local isothermal parametrization at every point.*

Theorem 3.0.1. *There exists no compact minimal surface.*

Proof.
Assume that a compact (i.e., closed and bounded) minimal surface S exists. Let $z : S \to \mathbb{R}$ be the height function. Being continuous on a compact set, z is bounded and attains its maximum and minimum.

Let $p \in S$ be such that $z(p)$ is maximal, and define:

$$Z = \{q \in S \mid z(q) = z(p)\}.$$

– Z is closed as the preimage of a closed set under a continuous function. – Let $q \in Z$. Consider a local isothermal parametrization $\Phi : U \to S$ near $q = \Phi(0, 0)$, with U connected.

Since S is minimal, by Corollary 2 the coordinates are harmonic. In particular, z is harmonic on U and bounded above by $z(0, 0)$. A bounded harmonic function on a convex open set is constant, so z is constant on U, hence $\Phi(U) \subset Z$, and Z is open.

Therefore, $Z = S$, and S lies in the plane $z = z(p)$. By repeating the argument with the x- and y-coordinates, we find that S lies in a single point, which contradicts the regularity of the surface.

□ □

Surfaces of Revolution

Definition 3.0.5.

Let $S \subset \mathbb{R}^3$ be the subset obtained by rotating a regular planar curve C around an axis in the plane of the curve.

We consider the curve C in the xz-plane and rotate it around the z-axis. Let $x = \varphi(v)$, $z = \rho(v)$, with $a < v < b$ and $\varphi(v) > 0$, be a parametrization of C. Then we define the surface via:

$$\Phi_1 : (u,v) \in (0, 2\pi) \times (a,b) \mapsto (\varphi(v)\cos(u),\ \varphi(v)\sin(u),\ \rho(v)),$$
$$\Phi_2 : (u,v) \in (-\pi, \pi) \times (a,b) \mapsto (\varphi(v)\cos(u),\ \varphi(v)\sin(u),\ \rho(v)).$$

These two local parametrizations cover the surface.
We now verify that the result is indeed a regular surface:

- Φ_1 and Φ_2 are C^∞ on their domains of definition.
- Let $p = (u,v) \in S$. Then the differential is:
$$d_p\Phi_i = \begin{pmatrix} -\varphi(v)\sin(u) & \varphi'(v)\cos(u) \\ \varphi(v)\cos(u) & \varphi'(v)\sin(u) \\ 0 & \rho'(v) \end{pmatrix},$$
which is injective since the curve C is regular.
- We must now prove that the map is injective (so that we can use the global inversion theorem and conclude it's a diffeomorphism). Assume $\Phi_1(u,v) = \Phi_1(u',v')$. Then:
$$\varphi(v)\cos(u) = \varphi(v')\cos(u'), \quad \varphi(v)\sin(u) = \varphi(v')\sin(u'), \quad \rho(v) = \rho(v').$$
This implies $\varphi^2(v) = \varphi^2(v')$, and since $\varphi(v) > 0$, we get $\varphi(v) = \varphi(v')$. Thus $(\varphi(v), \rho(v)) = (\varphi(v'), \rho(v'))$. Since the curve is injective, $v = v'$. Therefore, $u \equiv u' \mod 2\pi$, and within our parametrizations, this implies $u = u'$. Hence, the map is injective.

Definition 3.0.6.
A surface as described above is called a surface of revolution. *The curve C is called the* generating curve, *and z is the* axis of rotation *of S. The circles traced out by the points of C are called the* parallels *of S, and the different positions of C are called the* meridians *of S.*

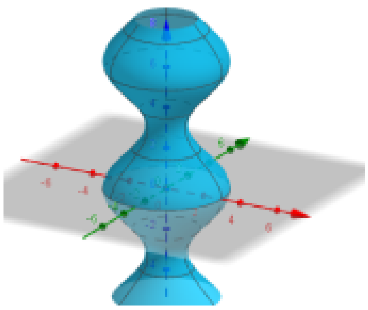


Figure 3.1

3.0.2 Associated Fundamental Forms

First Fundamental Form

Let us determine the coefficients of the matrix of the first fundamental form:
$$E = \langle \partial_u\Phi, \partial_u\Phi \rangle = \varphi^2, \quad F = \langle \partial_u\Phi, \partial_v\Phi \rangle = 0, \quad G = \langle \partial_v\Phi, \partial_v\Phi \rangle = (\varphi')^2 + (\rho')^2$$

We notice that if the generating curve is parameterized by arc length, then $G = 1$, which we assume from now on.

Second Fundamental Form
To determine the coefficients of the matrix of the second fundamental form, we first calculate the norm of $|\partial_u \Phi \wedge \partial_v \Phi|$:

$$|\partial_u \Phi \wedge \partial_v \Phi| = \begin{vmatrix} \varphi\cos(u) & 0 \\ \varphi\sin(u) & \rho' \\ -\varphi\varphi' & 0 \end{vmatrix} = \varphi\sqrt{(\rho')^2 + (\varphi')^2} = \varphi$$

Then, we have:

$$e = \langle N, \partial_{uu}\Phi \rangle = \frac{\det(\partial_u\Phi, \partial_v\Phi, \partial_{uu}\Phi)}{|\partial_u \Phi \wedge \partial_v \Phi|} = -\varphi\rho', \quad f = \langle N, \partial_{uv}\Phi \rangle = 0, \quad g = \langle N, \partial_{vv}\Phi \rangle = \rho'\varphi'' - \rho''\varphi'$$

3.0.3 Curvature

From the expressions of the coefficients and the known formulas for Gaussian curvature K and mean curvature H, we have:

$$K = \frac{eg}{EG}, \quad H = \frac{eG + gE}{2EG}$$

Hence:

$$K = -\frac{\rho'}{\varphi} \cdot \frac{\rho'\varphi'' - \rho''\varphi'}{\varphi} = -\frac{\varphi''}{\varphi}$$

and

$$H = \frac{-\rho + \varphi(\rho'\varphi'' - \rho''\varphi')}{2\varphi}$$

Now we will determine the lines of curvature of the surface. Indeed, the matrix of the differential $d_{(u,v)}N$ in the basis $\partial_u\Phi, \partial_v\Phi$ is given by:

$$\begin{pmatrix} -\frac{\rho'}{\varphi} & 0 \\ 0 & \frac{\rho''\varphi' - \rho'\varphi''}{\varphi} \end{pmatrix}$$

From the calculation of F, $\partial_u\Phi$ and $\partial_v\Phi$ are orthogonal. Thus, the principal curvatures are $-\frac{\rho'}{\varphi}$ and $\frac{\rho''\varphi' - \rho'\varphi''}{\varphi}$, and the principal directions are given by $\partial_u\Phi$ and $\partial_v\Phi$.

This also allows us to remark that the parallels and meridians are lines of curvature.

3.0.4 The Catenoid (Euler, 1744)

The catenoid was discovered by Leonard Euler in 1744 in order to answer the following question: what is the surface of minimal area connecting two parallel circles? The notion of minimal surface was later introduced by Lagrange in 1760.

The catenoid is the surface of revolution with the hyperbolic cosine as generating curve:

$$\Phi_c : (u, v) \mapsto \big(a\cosh(v)\cos(u),\ a\cosh(v)\sin(u),\ av\big)$$

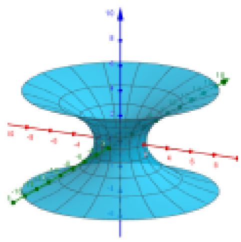


Figure 3.2- The Catenoid

Property 6.
The catenoid is a minimal surface.

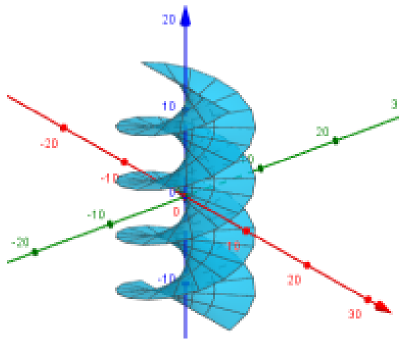
Proof.
From the previous calculations on surfaces of revolution, we have:

- $E_c = \langle \partial_u\Phi_c, \partial_u\Phi_c\rangle = a^2\cosh^2(v)$
- $G_c = \langle \partial_v\Phi_c, \partial_v\Phi_c\rangle = a^2\cosh^2(v)$
- $F_c = \langle \partial_u\Phi_c, \partial_v\Phi_c\rangle = 0$

Hence, the conditions of Proposition apply and the mean curvature H is zero at every point.

□ □

3.0.5 The Helicoid (Meusnier 1776)



The Helicoid (Meusnier 1776)

Definition 3.0.7.

The helicoid is the surface parametrized by:

$$\Phi_h : (u, v) \mapsto \big(a \sinh(v) \cos(u), \quad a \sinh(v) \sin(u), \quad au\big).$$

Proposition 3.0.4.
The helicoid is a minimal surface.

Proof.
We compute the partial derivatives:

$$\partial_u \Phi_h = \big(-a \sinh(v) \sin(u), \quad a \sinh(v) \cos(u), \quad a\big),$$

$$\partial_v \Phi_h = \big(a \cosh(v) \cos(u), \quad a \cosh(v) \sin(u), \quad 0\big).$$

The coefficients of the first fundamental form are:

$$E_h = \langle \partial_u \Phi_h, \partial_u \Phi_h \rangle = a^2 + a^2 \sinh^2(v) = a^2 \cosh^2(v),$$

$$G_h = \langle \partial_v \Phi_h, \partial_v \Phi_h \rangle = a^2 \cosh^2(v),$$
$$F_h = \langle \partial_u \Phi_h, \partial_v \Phi_h \rangle = 0.$$

Moreover,

$$\partial_{uu} \Phi_h = \big(-a \sinh(v) \cos(u), \quad -a \sinh(v) \sin(u), \quad 0\big),$$

$$\partial_{vv} \Phi_h = \big(a \sinh(v) \cos(u), \quad a \sinh(v) \sin(u), \quad 0\big).$$

Hence,
$$\partial_{uu} \Phi_h + \partial_{vv} \Phi_h = 0.$$

By property 19, the mean curvature H is thus zero everywhere, which implies that the helicoid is a minimal surface. □

3.1 Conjugate Minimal Surfaces

Definition 3.1.1 (Conjugate Harmonics and Conjugate Minimal Surfaces)**.**
Functions $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ *and* $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ *are said to be* conjugate harmonic functions *if they satisfy the Cauchy-Riemann equations:*

$$\partial_u f = \partial_v g, \quad \partial_v f = -\partial_u g.$$

It can be easily shown that such functions are harmonic.
Two minimal surfaces are called conjugate minimal surfaces *if their coordinate functions are pairwise conjugate harmonic functions.*

Proposition 3.1.1.
If α *and* β *are two conjugate minimal surfaces with isothermal parameters, then for any* $t \in \mathbb{R}$*,*

$$\gamma = \cos(t)\,\alpha + \sin(t)\,\beta$$

is also a minimal surface.

Proof.
Using that the surfaces are conjugate and have isothermal parameters, we obtain:

$$E_\gamma = \cos^2(t)E_\alpha + \sin^2(t)E_\beta + 2\cos(t)\sin(t)\langle \partial_u\alpha, \partial_u\beta\rangle = E_\alpha,$$

$$F_\gamma = \cos^2(t)F_\alpha + \sin^2(t)F_\beta + \cos(t)\sin(t)\big(\langle \partial_u\alpha, \partial_v\beta\rangle + \langle \partial_v\alpha, \partial_u\beta\rangle\big) = 0,$$

$$G_\gamma = \cos^2(t)G_\alpha + \sin^2(t)G_\beta + 2\cos(t)\sin(t)\langle \partial_v\alpha, \partial_v\beta\rangle = E_\alpha.$$

Thus, the coordinates of γ are isothermal. We still need to check that they are harmonic:

$$\partial_{uu}\gamma + \partial_{vv}\gamma = \cos(t)(\partial_{uu}\alpha + \partial_{vv}\alpha) + \sin(t)(\partial_{uu}\beta + \partial_{vv}\beta).$$

Since α and β are isothermal and minimal, their coordinates are harmonic, hence

$$\partial_{uu}\gamma + \partial_{vv}\gamma = 0.$$

Therefore, γ is indeed a minimal surface.

□

Remark 3.1.1.
This shows that once two conjugate minimal surfaces are found, one can construct infinitely many minimal surfaces. Moreover, two conjugate minimal surfaces are locally isometric.

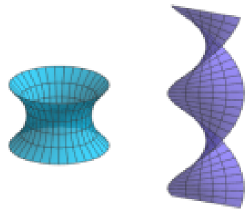
3.2 Associated Family of the Catenoid and Helicoid

The helicoid was discovered by Meusnier in 1776 and, along with the catenoid, remained for a long time one of the only known minimal surfaces. They satisfy:

$$E_c = E_h, \quad F_c = F_h, \quad G_c = G_h.$$

Consider Φ_c and Φ_h defined on $(0, 2\pi) \times \mathbb{R}$, then we have parametrizations of the catenoid deprived of a meridian and of the "first upper torsion" of the helicoid.

From Proposition , $\Phi_c \circ \Phi_h^{-1}$ and $\Phi_h \circ \Phi_c^{-1}$ are isometries between the above surfaces. Hence, the helicoid and catenoid are locally isometric.

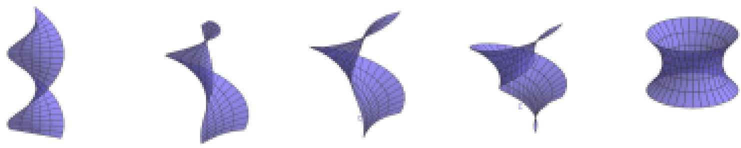


Catenoid Helicoid

Let us now show that these surfaces are conjugate:

$$\begin{cases} \partial_u x_c = -a\sin(u)\cosh(v), & \partial_u x_h = -a\sin(u)\sinh(v), \\ \partial_u y_c = a\cos(u)\cosh(v), & \partial_u y_h = a\cos(u)\sinh(v), \\ \partial_u z_c = 0, & \partial_u z_h = a, \\ \partial_v x_c = a\sinh(v)\cos(u), & \partial_v x_h = a\cosh(v)\cos(u), \\ \partial_v y_c = a\sinh(v)\sin(u), & \partial_v y_h = a\cosh(v)\sin(u), \\ \partial_v z_c = a, & \partial_v z_h = 0. \end{cases}$$

Thus, the following pairs form conjugate harmonic function couples: (y_h, x_c), (y_c, x_h), and (z_h, z_c).



Families of Minimal Surfaces Associated with the Catenoid and the Helicoid

We conclude that the catenoid and helicoid are conjugate minimal surfaces, yielding an associated family of minimal surfaces.

3.3 More Complex Surfaces

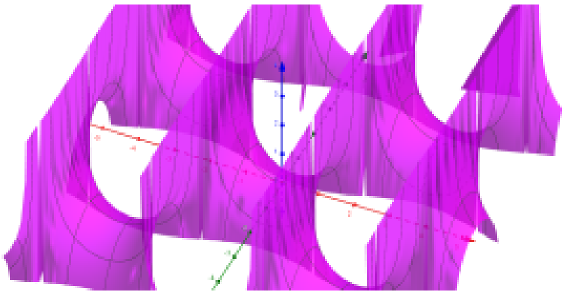
Until 1835, the two minimal surfaces described above were the only known ones. We will now study other minimal surfaces.

3.3.1 Scherk's Surface (1835)

We study the surface parametrized by:

$$\Phi_s : (u, v) \mapsto \left(u, v, \log\left(\frac{\cos(v)}{\cos(u)}\right)\right).$$

This parametrization is not Scherk's original, but it simplifies calculations.



Scherk's Surface (1835)

Derivatives

$$\partial_u \Phi_s = (1, 0, \tan(u)), \quad \partial_v \Phi_s = (0, 1, -\tan(v)), \quad \partial_{uv} \Phi_s = (0, 0, 0),$$
$$\partial_{uu} \Phi_s = (0, 0, 1 + \tan^2(u)), \quad \partial_{vv} \Phi_s = (0, 0, -(1 + \tan^2(v))).$$

First Fundamental Form

$$E_s = \langle \partial_u \Phi_s, \partial_u \Phi_s \rangle = 1 + \tan^2(u),$$
$$F_s = \langle \partial_u \Phi_s, \partial_v \Phi_s \rangle = 0,$$
$$G_s = \langle \partial_v \Phi_s, \partial_v \Phi_s \rangle = 1 + \tan^2(v).$$

Gauss Map

The unit normal vector is

$$N = \frac{\partial_u \Phi_s \wedge \partial_v \Phi_s}{|\partial_u \Phi_s \wedge \partial_v \Phi_s|} = \frac{(-\tan(u), \tan(v), 1)}{\sqrt{1 + \tan^2(u) + \tan^2(v)}}.$$

Second Fundamental Form

$$e_s = \langle N, \partial_{uu}\Phi_s \rangle = \frac{1+\tan^2(u)}{\sqrt{1+\tan^2(u)+\tan^2(v)}},$$
$$f_s = \langle N, \partial_{uv}\Phi_s \rangle = 0,$$
$$g_s = \langle N, \partial_{vv}\Phi_s \rangle = \frac{-(1+\tan^2(v))}{\sqrt{1+\tan^2(u)+\tan^2(v)}}.$$

Minimal Surface Condition

Calculate:

$$e_sG_s + g_sE_s - 2f_sF_s = \frac{(1+\tan^2(u))(1+\tan^2(v)) - (1+\tan^2(v))(1+\tan^2(u))}{\sqrt{1+\tan^2(u)+\tan^2(v)}} - 0 = 0.$$

Hence, the mean curvature is zero:

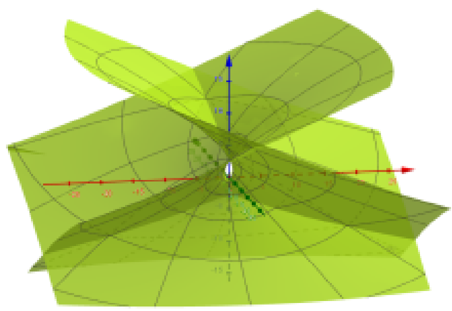
$$H = \frac{e_sG_s + g_sE_s - 2f_sF_s}{2(E_sG_s - F_s^2)} = 0.$$

3.3.2 Enneper's Surface (1863)

The Enneper surface is parametrized by:

$$\Phi_e : (u,v) \mapsto \left(u - \frac{u^3}{3} + uv^2, \quad v - \frac{v^3}{3} + vu^2, \quad u^2 - v^2 \right).$$

As noted earlier, a regular parametrized surface may have self-intersections, which is the case here.



Enneper's Surface (1863)

Derivatives

$$\partial_u \Phi_e = \big(1 - u^2 + v^2, \quad 2uv, \quad 2u\big), \quad \partial_v \Phi_e = \big(2uv, \quad 1 - v^2 + u^2, \quad -2v\big),$$

$$\partial_{uv} \Phi_e = (2v, \quad 2u, \quad 0), \quad \partial_{uu} \Phi_e = (-2u, \quad 2v, \quad 2), \quad \partial_{vv} \Phi_e = (2u, \quad -2v, \quad -2).$$

First Fundamental Form

$E_e = \langle \partial_u \Phi_e, \partial_u \Phi_e \rangle = (1+u^2+v^2)^2, \quad F_e = \langle \partial_u \Phi_e, \partial_v \Phi_e \rangle = 0, \quad G_e = \langle \partial_v \Phi_e, \partial_v \Phi_e \rangle = (1+u^2+v^2)^2.$

Minimal Surface Condition By proposition , the surface is minimal if and only if

$$\partial_{uu} \Phi_e + \partial_{vv} \Phi_e = 0.$$

From the above derivatives, this relation is verified, thus Enneper's surface is minimal.

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