UNIVERSITY CENTER OF NAÂMA, SALHI AHMED

MASTER MEMORY

Existence Of Solution For Nonlinear Ordinary Differential equations

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Abstract

Faculty of Science and Technology
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Master of Mathematics

Existence Of Solution For Nonlinear Ordinary Differential equations

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In this memory, we are studying the existence of solutions for nonlinear fourth-order boundary value problem. We give sufficient conditions that allow us to obtain the existence of solution. By using the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem, we prove the existence of positive solution of the posed problem in chapter 2. We prove also the existence at least one nontrivial solution for boundary value problem, the main tool used in the proof is the Leray-Schauder nonlinear alternative in the last chapter 3.

Key words: Existence of solution, Positive solutions, Solvability, Green's function, Nontrivial solution, Leary-Schauder nonlinear alternative, Ascoli-Arzela theorem, Fixed point theorem, Boundary value problem.

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Amina BENHAMMOU.

To my dear parents, May God keep them.
To all my brothers and sisters.
To all who are close to my heart,
and whose names I did not mention.
All my friends.

Contents

List of Symbols

$\Omega = bounded\ \ open\ \ subset\ \ of\ \ E.$

$C(\Omega) = \{u :\ \Omega \longrightarrow \mathbf{R},\ \ u\ has\ continuous\ functions\ on\ \Omega\}.$

$\|f\|_\infty = \max_{x\in\Omega} |f(x)|, \quad is\ uniform\ norm\ of\ f.$

$C^k(\Omega) = \{f \in C^k(\Omega),\ \ \frac{\partial f}{\partial x_i} \in C^{k-1}(\Omega),\ \ i = 1, ..., n\},\ \ k \geq 1.$

$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbf{R},\ \ u\ is\ measurable\ and\ \ \int_\Omega |u(x)|^p dx < \infty\}, 1 \leq p < +\infty.$

$\|u\|_p = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}, \quad 1 \leq p < +\infty$

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbf{R} \;\middle|\; \begin{array}{l} u\ is\ measurable\ and\ there\ exists\ a\ constant\ C > 0 \\ such\ that, |u(x)| \leq C\ \ a.e\ \ in\ \Omega \end{array} \right\},$$

$\|u\|_\infty = \inf\{C, |u| \leq C\ \ a.e\ \ on\ \Omega\}, \quad .$

$\{fm\}_{m\in\mathbf{N}} is\ sequence\ of\ f.$

$B_E : the\ closed\ unit\ ball\ of\ E.$

$BVP : \quad Boundary\ Value\ Problem.$

$FBVP : \quad Fourth - Order\ Boundary\ Value\ Problem.$

General Introduction

Boundary value problems for ordinary differential equation play a very important role in both theory and applications. They are used to describe a large number of physical sciences, engineering, biological and chemical phenomena. For examples, a second-order three-point (BVP) is used as a model for the membrane of a spherical cap in nonlinear diffusion generated by nonlinear sources and in chemical reactor theory, and third-order three-point boundary value problem is represented in population dynamics, the process of heat conduction, control theory, and we find it also in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows, see ([13], [20]). Also the deformations of an elastic beam are described by a fourth-order differential equation, often referred to as the beam equation ([17]), for Some other results on fourth-order boundary value problem to study the existence or nonexistence or positivity of solution by using several different methods for example fixed point theorems in cones, Leray-Schauder nonlinear alternative, Leray-Schauder fixed point theorem, and Krasnoselskii's fixed point theorem, we refer the reader to the papers ([3], [4], [7], [8], [11], [12], [14], [18], [19], [23]).

This work consists of three chapters.

- In **the chapter 1.**

  In this chapter which is represented in a general introduction that includes all preliminaries and materials needed, we given basic definitions, important notions, lemmas, and theorems used in this work. We have divided this chapter into two main parts.

  1. Functional analysis.
  2. Fixed-point theorems.

     See ([1], [5], [6], [9], [10], [15], [16], [21], [22]) and references there is for more details.

- In **the chapter 2.**

  In this chapter, derived from ([17]), we consider the fourth-order boundary value problem of the form

$$\begin{aligned} &u^{(4)}(t) = q(t)f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \\ &u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = 0 \end{aligned} \tag{1}$$

  where $q : [0.1] \longrightarrow [0.\infty)$, $f : [0.1] \times [0.\infty) \times [0.\infty) \times (-\infty.0] \times (-\infty.0] \longrightarrow [0.\infty)$ are continuous, motivated by this work, we investigate the existence of positive solutions for fourth-order boundary value problem

(FBVP) (1), we give sufficient conditions that allow us to obtain the existence of least one positive solution of SBVP (1), by using main tool in the proof which are the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem. In the main section of this chapter, we assume that $q(t) \equiv 1$ and the corresponding Green's function is nonnegative, we present our main results which consists of two basic theorems, theorem $2.2.2$ and theorem $2.2.4$ to prove the existence of at least one positive solution to the FBVP (1). Finally, as an application, we give an example to illustrate the results we obtained in theorem $2.2.2$.

- In **the chapter 3.**

  In this chapter, derived from ([2]), we consider the fourth-order three-point boundary value problem having the following form

$$\begin{aligned} &u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = \alpha u'(\eta), \end{aligned} \tag{2}$$

  where $\eta \in (0,1)$, $\alpha \in \mathbf{R}$, $\alpha \neq 1$, $f \in C([0,1] \times \mathbf{R}, \mathbf{R})$.

  To study this boundary value problem (BVP) (2), we used Leray-Schauder nonlinear alternative to prove the existence of solution of the posed problem. We adopted some theorems and corollary for prove the existence of least one nontrivial solution of the BVP (2) and we set all sufficient conditions that allow us to obtain the existence of solution. In theorem $3.3.1$, we assumed that the conditions $f(t,0) \neq 0$, $\alpha \neq 1$, $|f(t,x)| \leq k(t)|x| + h(t)$, and $M < 1$ such that M is a positive constant, to prove that the BVP (2) has at least one nontrivial solution. In theorem $3.3.2$, we assumed the same conditions of the previous theorem with changing condition $\alpha < 1$ and formula for M, under the light these data we established four conditions to prove that BVP (2) has at least one nontrivial solution. Then in theorem $3.3.3$, we took the same conditions with adding a change on $\alpha > 1$ and on formula the M, depending on these changes we have assumed four new conditions to prove the same existence of the above problem. In corollary $3.3.4$, we take the same conditions of the theorem $3.3.2$ with changing the format M by assuming a value $\eta = 1$, of these assumptions we offer three conditions to prove the existence of at least one nontrivial solution of the BVP (2). In corollary $3.3.5$, We are based on the conditions mentioned in theorem $3.3.3$ with taking a new formula of M by assuming a value $\eta = 1$, depending on these data we established also three conditions to prove the existence of solution of the posed problem. Finally, we give some examples to illustrate the results obtained in theorem $3.3.1$ and theorem $3.3.2$.

Chapter 1

Preliminaries

All assertions in the first chapter are made without proofs and the scope has been minimized to only material actually needed.

1.1 Functional Analysis

1.1.1 Hölder's inequality

Notation: Let $1 < p \leq \infty$, we denote by q the conjugate exponent,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$ and Ω is a bounded open subset. Then $(fg) \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q.$$

1.1.2 Normal Cones

Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$,

(ii) $x \in P, -x \in P$ implies $x = 0$, where 0 denotes the zero element E.

Every cone P in E and for all $x, y \in P$ defines a partial ordering in E given by

$$x \leq y \quad if\ and\ only\ if \quad y - x \in P.$$

Definition

Suppose P is a cone in a Banach space E. The map ψ is a nonnegative continuous concave functional on P provided $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say the map φ is a nonnegative continuous convex functional on P provided $\varphi : P \to [0, \infty)$ is continuous and

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$

1.1.3 Non-Linear Operator

A mapping of a space (as a rule, a vector space) X into a vector space Y over a common field of scalars that does not have the property of linearity, that is, such that generally speaking

$$A(\alpha_1 x_1 + \alpha_2 x_2) \neq \alpha_1 A x_1 + \alpha_2 A x_2.$$

If Y is the set $\mathbf{R}$ of real or $\mathbf{C}$ of complex numbers, then a non-linear operator is called a non-linear functional.

1.1.4 Compact Operators

A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if $T(B_E)$ has compact closed in F .

The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, E)$.

1.1.5 Operator Completely Continuous

An operator $T : E \longrightarrow E$ is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

1.1.6 Convex Sets

Let $u, v \in V$. Then the set of all convex combinations of u and v is the set of points

$$\{\forall \lambda \in [0.1] : \quad w = (1 - \lambda)u + \lambda v, \;\; 0 \leq \lambda \leq 1\}.$$

Next, is the notion of a convex set.

1.1.7 Precompact Set

A subset in a topological space is precompact if its closed is compact.

1.2 Fixed-Point Theorems

1.2.1 Schauder Fixed Point Theorem

Let X be a Banach space and let $K \subset X$ be a compact and convex subset. If $f : K \longrightarrow K$ is continuous, then f has a fixed point.

Corollary

Let K be a closed, bounded and convex subset of X and let $f : K \longrightarrow K$ be compact. Then f has a fixed point.

1.2.2 Leray-Schauder Nonlinear Alternative

Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $T : \overline{\Omega} \rightarrow E$ be a completely continuous operator. Then, either

(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $T(u) = \lambda u$, or

(ii) there exists a fixed point $u^* \in \overline{\Omega}$ of T.

1.2.3 Theorem (Leray-Schauder Nonlinear Alternative)

Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $p \in U$. Suppose that $F : \overline{U} \longrightarrow C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either

$(A1)$ F has a fixed point in $\overline{U}$, or

$(A2)$ there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)p$.

1.2.4 Arzela-Ascoli Theorem

A subset M of $C([a, b], \mathbf{R}^n)$ is relatively compact if and only if it is bounded and equicontinuous.

1.2.5 Arzela-Ascoli Theorem

If a sequence $\{fm\}_{m\in\mathbf{N}}$ in $C(K)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.

1.2.6 Arzela-Ascoli Theorem

Let (K, d) be a compact metric space, $(E, \|.\|)$ a Banach space and $A \subset C(K, E)$. Then A is relatively compact in $(C(K, E), \|.\|_{\infty,K})$ if and only if the two conditions below are satisfied:

(a) A is equicontinuous, ie, for all $x \in K$ and for all $\epsilon > 0$ there is a neighborhood $V \subset K$ of x such that $\|f(x) - f(y)\| < \epsilon$, $\forall y \in V$, $\forall f \in A$,

(b) $A(x) := \{f(x),\ \ f \in A\}$ is relatively compact in E.

Chapter 2

Existence Of Positive Solution For Nonlinear Ordinary Differential equations

2.1 Introduction

In this chapter, we study the existence of positive solutions for the fourth-order boundary value problem of the form

$$u^{(4)}(t) = q(t)f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \tag{2.1}$$

and the boundary conditions

$$u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = 0 \tag{2.2}$$

where $q : [0.1] \longrightarrow [0.\infty)$, $f : [0.1] \times [0.\infty) \times [0.\infty) \times (-\infty.0] \times (-\infty.0] \longrightarrow [0.\infty)$ are continuous, we give sufficient conditions that allow us to obtain the existence of positive solution. The main tool used in the proof is the Leray-Schauder nonlinear alternative and Leray-Schauder fixed point theorem. As an application, we also given an example to illustrate the results obtained.

2.2 Mains results

In this section, we have given some lemmas, theorems, and we shall impose growth sufficient conditions on f which allow us to apply Leray-Schauder nonlinear alternative and Leray-Schauder theorem to establish the existence of at least one positive solution to problem $(2.1), (2.2)$. We assume that $q(t) \equiv 1$.

2.2.1 Lemma

Let $E = \{u \in C^3[0,1] : u(0) = u'(1) = u''(0) = 0\}$ be the Banach space equipped with the maximum norm

$$||u|| = max\{|u|_0, |u'|_0, |u''|_0, |u'''|_0\}$$

$|u|_0 = max_{0 \leq t \leq 1}|u(t)|$, Then for any $u \in E$, we have

$$||u|| = |u'''|_0 \ , \ |u|_0 \leq \frac{1}{3}||u|| \ , \ |u'|_0 \leq \frac{1}{2}||u|| \ and \ |u''|_0 \leq ||u||$$

if $||u|| = |u^{(3)}|_0$

Proof. Let $H(t,s)$ be the Green's function of third-order homogeneous boundary value problem

$$-u'''(t) = 0, \quad t \in [0,1]$$

$$u(0) = u'(1) = u''(0) = 0,$$

Then

$$H(t,s) = \begin{cases} \frac{1}{2}(2t - t^2 - s^2), & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

First, we shall determine the Green's function. We have

$$-u'''(t) = 0, \ \ for \ \ t \ \in [0,1], \ \ with \ \ u(0) = u'(1) = u''(0) = 0.$$

Integrating from 0 to 1, we get

$$u(t) = \alpha + \beta t + \gamma t^2, \quad avec \ \ \alpha, \beta, \gamma \in \mathrm{R}$$

Then the Green function is the form

$$H(t,s) = \begin{cases} a_1 + a_2 t + a_3 t^2, & 0 \le t \le s \le 1, \\ b_1 + b_2 t + b_3 t^2, & 0 \le s \le t \le 1, \end{cases}$$

with $a_1, a_2, a_3, b_1, b_2, b_3$ are continuous functions of s.

So, the boundary conditions, we obtain

$$H(0.s) = H''(0.s) = 0$$

i.e.

$$a_1 = a_3 = 0$$

and

$$H'(1.s) = 0$$

i.e.

$$b_2 + 2b_3 = 0,$$

we pose $c_k = b_k - a_k, (k = 1,2,3)$, and the system for linear equations

$$\begin{cases} c_1 + c_2 s + c_3 s^2 = 0, \\ c_2 + 2c_3 s = 0, \\ c_3 = -1 \end{cases}$$

implies

$$c_1 = -\frac{1}{2}s^2, \quad c_2 = s, \quad c_3 = -\frac{1}{2},$$

and

$$a_2 = 1 - s, \quad b_1 = -\frac{1}{2}s^2, \quad b_2 = 1, \quad b_3 = -\frac{1}{2}.$$

Then the Green function is given by

$$H(t,s)=\begin{cases}\frac{1}{2}(2t-t^2-s^2), & 0\le s\le t\le 1,\\ (1-s)t, & 0\le t\le s\le 1,\end{cases}$$

Thus

$$\frac{\partial}{\partial t}H(t,s)=\begin{cases}(1-t), & 0\le s\le t\le 1,\\ (1-s), & 0\le t\le s\le 1,\end{cases}$$

$$\frac{\partial^2}{\partial t^2}H(t,s)=\begin{cases}-1, & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1,\end{cases}$$

By (2.3) it is easy to know that

$$H(t,s)\ge 0,\quad \frac{\partial}{\partial t}H(t,s)\ge 0,\quad \frac{\partial^2}{\partial t^2}H(t,s)\le 0 \tag{2.4}$$

and

$$\int_0^1|H(t,s)|ds=\int_0^1 H(t,s)ds=-\frac{1}{6}t^2+\frac{1}{2}t$$

$$\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right|ds=\int_0^1\frac{\partial}{\partial t}H(t,s)ds=-\frac{1}{2}t^2+\frac{1}{2}$$

$$\int_0^1\left|\frac{\partial^2}{\partial t^2}H(t,s)\right|ds=-\int_0^1\frac{\partial^2}{\partial t^2}H(t,s)ds=t.$$

From which we get

$$\max_{0\le t\le 1}\int_0^1|H(t,s)|ds=\frac{1}{3},\quad \max_{0\le t\le 1}\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right|ds=\frac{1}{2},$$

$$\max_{0\le t\le 1}\int_0^1\left|\frac{\partial^2}{\partial t^2}H(t,s)\right|ds=1.$$

Let $u\in E$ and $||u||=\rho$,

$$u(t)=\int_0^1 H(t,s)[-u'''(s)]ds,\quad u'(t)=\int_0^1\frac{\partial}{\partial t}H(t,s)[-u'''(s)]ds,$$

$$u''(t)=\int_0^1\frac{\partial^2}{\partial t^2}H(t,s)[-u'''(s)]ds$$

Thus

$$|u|_0\le \max_{0\le t\le 1}\int_0^1|H(t,s)||u'''(s)|ds\le |u'''|_0\max_{0\le t\le 1}\int_0^1|H(t,s)|ds=\frac{1}{3}|u'''|_0$$

$$|u'|_0\le \max_{0\le t\le 1}\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right||u'''(s)|ds\le |u'''|_0\max_{0\le t\le 1}\int_0^1\left|\frac{\partial}{\partial t}H(t,s)\right|ds=\frac{1}{2}|u'''|_0$$

$$|u''|_0\le \max_{0\le t\le 1}\int_0^1\left|\frac{\partial^2}{\partial t^2}H(t,s)\right||u'''(s)|ds\le |u'''|_0\max_{0\le t\le 1}\int_0^1\left|\frac{\partial^2}{\partial t^2}H(t,s)\right|ds=|u'''|_0$$

So, $|u^{(3)}|_0=||u||=\rho$, and the proof is completed

2.2.2 Theorem

Suppose that $f \in C([0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0] \times (-\infty,0],[0,+\infty))$ and $f(t,0,0,0,0) \not\equiv 0$, $t \in [0,1]$. Suppose there exist nonnegative functions $a_i \in L^1[0,1]$, $i = 0,1,2,3,4$, such that

$$B = \frac{1}{3}\int_0^1 a_0(s)ds + \frac{1}{2}\int_0^1 a_1(s)ds + \int_0^1 a_2(s)ds + \int_0^1 a_3(s)ds < 1, \qquad (2.5)$$

and for any $(t,\alpha_0,\alpha_1,\alpha_2,\alpha_3) \in [0,1] \times [0,\frac{\rho}{3}] \times [0,\frac{\rho}{2}] \times [-\rho,0] \times [-\rho,0]$, f satisfies

$$f(t,\alpha_0,\alpha_1,\alpha_2,\alpha_3) \leq a_0(t)\alpha_0 + a_1(t)\alpha_1 - a_2(t)\alpha_2 - a_3(t)\alpha_3 + a_4(t), \qquad (2.6)$$

where $\rho = A(1-B)^{-1}$, $A = \int_0^1 a_4(s)ds$ .Then problem $(2.1),(2.2)$ has at least one positive solution $u^* \in C^4([0,1])$ such that

$$3\max_{0\leq t\leq 1} u^*(t) \leq 2\max_{0\leq t\leq 1}(u^*)'(t) \leq \max_{0\leq t\leq 1}[-(u^*)''(t)] \leq \max_{0\leq t\leq 1}[-(u^*)'''(t)] \leq \rho.$$

Proof. Since $f(t,0,0,0,0) \not\equiv 0$ and $|f(t,0,0,0,0)| \leq a_4(t), t \in [0,1]$, we have $A = \int_0^1 a_4(s)ds > 0$, so, it follows from (2.5) that $\rho > 0$. From equation (2.1) and boundary condition $u^{(3)}(1) = 0$, we have

$$u^{(3)} = -\int_t^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau,$$

which implies that

$$u(t) = \int_0^1 H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds, \quad t \in [0,1],$$

where $H(t,s)$ is defined by (2.3). Let $\Omega_\rho = \{u \in E : ||u|| \leq \rho\}$, then Ω_ρ is a bounded closed convex set of E and $0 \in \Omega_\rho$. For $u \in \Omega_\rho$, define the operator T by

$$(Tu)(t) = \int_0^1 H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds, \quad t \in [0,1] \qquad (2.7)$$

Then

$$(Tu)'(t) = \int_0^1 \frac{\partial}{\partial t}H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds, \quad t \in [0,1],$$

$$(Tu)''(t) = \int_0^1 \frac{\partial^2}{\partial t^2}H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds, \quad t \in [0,1],$$

$$(Tu)'''(t) = -\int_t^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau, \quad t \in [0,1].$$

So, $(Tu)(0) = (Tu)'(1) = (Tu)''(0) = (Tu)'''(1) = 0$. Therefore, $T : \Omega_p \longrightarrow E$. By Ascoli-Arzela Theorem, it is easy to know that this operator $T : \Omega_p \longrightarrow E$ is a completely continuous operator. So, the problem $(2.1),(2.2)$ has a solution $u = u(t)$ if and only if u solves the operator equation $Tu = u$.

Let us prove T is completely continuous, we have $\Omega_\rho = \{u \in E : ||u|| \leq \rho\}$ is a bounded closed convex set of E. We shall prove that $T(\Omega_\rho)$ is relatively compact. The proof will be done is some steps.

(i) Let $u \in \Omega_\rho$, we have by (2.7)

$$|Tu(t)| = |\int_0^1 H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds|$$

$$\leq \int_0^1 |H(t,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau|ds$$

$$\leq \int_0^1 |H(t,s)|\int_s^1 |f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))|d\tau ds$$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)| \max_{0\leq s\leq 1}\int_s^1 |f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))|d\tau ds$$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|\int_0^1 |f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))|d\tau ds$$

According to (2.6), we find $|Tu(t)|$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|\int_0^1 |a_0(t)u(\tau)+a_1(t)u'(\tau)-a_2(t)u''(\tau)-a_3(t)u'''(\tau)+a_4(t)|d\tau ds$$

By Lemma 2.2.1, we get $|Tu(t)|$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|\int_0^1 a_0(t)|u|_0+a_1(t)|u'|_0+a_2(t)|u''|_0+a_3(t)|u'''|_0+a_4(t)d\tau ds$$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|\int_0^1 (\frac{1}{3}a_0(t)\|u\|+\frac{1}{2}a_1(t)\|u\|+a_2(t)\|u\|+a_3(t)\|u\|+a_4(t))d\tau ds$$

Let $u \in E$ and $\|u\| = \rho$. Then $|Tu(t)|$

$$\leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|((\frac{1}{3}\int_0^1 a_0(\tau)d\tau+\frac{1}{2}\int_0^1 a_1(\tau)d\tau+\int_0^1 a_2(\tau)d\tau+\int_0^1 a_3(\tau)d\tau)\rho$$
$$+\int_0^1 a_4(\tau))d\tau ds$$

By (2.5), we obtain

$$|Tu(t)| \leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|(B\rho+\int_0^1 a_4(\tau)d\tau)ds$$

$$= \max_{0\leq t\leq 1}\int_0^1 |H(t,s)(B\rho+A)ds$$

We have $\rho = A(1-B)^{-1}$, so

$$|Tu(t)| \leq \max_{0\leq t\leq 1}\int_0^1 |H(t,s)|(B\rho+(1-B)\rho)ds$$

$$= \rho\max_{0\leq t\leq 1}\int_0^1 |H(t,s)|ds$$

Therefore

$$|Tu(t)| \leq \frac{\rho}{3}.$$

Consequently $T((\Omega_\rho)$ is uniformly bounded.

(ii) Let us prove that $T((\Omega_\rho)$ is equicontinuous. Let $t_1, t_2 \in [0,1], t_1 < t_2, u \in \Omega_\rho$. We have

$$|Tu(t_1) - Tu(t_2)| = |\int_0^1 H(t_1,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds-$$

$$\int_0^1 H(t_2,s)\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds|$$

$$|Tu(t_1)-Tu(t_2)| = |\int_0^1 (H(t_1,s)-H(t_2,s))\int_s^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau ds|$$

Thus from (2.3), we have

$$|Tu(t_1)-Tu(t_2)| \leq \Big[\int_0^{t_1}\frac{1}{2}(2t_1-t_1^2-s^2)ds+\int_{t_1}^1(1-s)t_1 ds\int_0^{t_2}\frac{1}{2}(2t_2-t_2^2-s^2)ds$$

$$+\int_{t_2}^1(1-s)t_2 ds\Big]\int_0^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau$$

Therfore,

$$|Tu(t_1)-Tu(t_2)| \leq \Big[\int_{t_1}^{t_2}\frac{1}{2}[2(t_1-t_2)-(t_1^2-t_2^2)]ds+\int_{t_1}^{t_2}(1-s)(t_1-t_2)ds\Big]$$

$$\int_0^1 f(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau))d\tau$$

Letting $t_1 \mapsto t_2$, then $|Tu(t_1)-Tu(t_2)|$ tends to 0. Consequently $T(\Omega_\rho)$ is equicontinuous, we deduce that T is completely continuous.

Suppose there exists $u \in \partial\Omega_\rho$, $\lambda > 1$ such tat $Tu = \lambda u$ .Noticing that $||u|| = \rho$,it follows from Lemma $(2.2.1)$ that

$$|u_0| \leq \frac{1}{3}\rho, \quad |u_0'| \leq \frac{1}{2}\rho, \quad |u_0''| \leq \rho, \quad |u_0'''| = \rho.$$

Thus from $(2.5), (2.6)$ and (2.7), we have

$$\begin{aligned}
\lambda\rho &= \lambda||u|| = ||Tu|| = \max_{0\leq t\leq 1}|u'''(t)| \\
&= \max_{0\leq t\leq 1}\Big|-\int_t^1 f(s,u(s),u'(s),u''(s),u'''s))ds\Big| \\
&= \max_{0\leq t\leq 1}\int_t^1 f(s,u(s),u'(s),u''(s),u'''s))ds \\
&= \int_0^1 f(s,u(s),u'(s),u''(s),u'''s))ds \\
&\leq \int_0^1\Big[a_0(s)u(s)+a_1(s)u'(s)-a_2(s)u''(s)-a_3(s)u'''(s)+a_4(s)\Big]ds \\
&\leq \int_0^1\Big[\frac{1}{3}a_0(s)\rho+\frac{1}{2}a_1(s)\rho+a_2(s)\rho+a_3(s)\rho+a_4(s)\Big]ds \\
&= \Big[\frac{1}{3}\int_0^1 a_0(s)ds+\frac{1}{2}\int_0^1 a_1(s)ds+\int_0^1 a_2(s)ds+\int_0^1 a_3(s)ds\Big]\rho+\int_0^1 a_4(s)ds
\end{aligned}$$

$$= \quad B\rho + A = B\rho + (1-B)\rho = \rho.$$

a contradiction. So, by Theorem $(2.2.2)$, T has a fixed point $u^* \in E$ which is a solution of the problem $(2.1),(2.2)$. Noticing that $f(t,0,0,0,0) \not\equiv 0$, we assert that $u = 0$ is not a solution of the problemn$(2.1),(2.2)$,therefore $|u^*|_0 > 0$ It follows from (2.4) that $u^*(t)$ is nondecreasing and concave on $[0,1]$, thus $u^*(t) \geq t|u^*|_0 > 0$ for $t \in [0,1]$, i.e., $u^*(t)$ is a positive solution of the problem $(2.1),(2.2)$. This completes proof.

2.2.3 Lemma

The Green's function of the fourth-order homogeneous equation $u''''(t) = 0$, $t \in [0,1]$, with boundary condition (2.2) is

$$G(t,s) = \frac{1}{6}\begin{cases} (6t-3t^2-s^2)s, & 0 \leq s \leq t \leq 1, \\ (6s-3s^2-t^2)t,, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.8}$$

and for any $t,s \in [0.1]$,

$$G(t,s) \geq 0 \ , \ \frac{\partial}{\partial t}G(t,s) \geq 0 \ , \ \frac{\partial^2}{\partial t^2}G(t,s) \leq 0 \ , \ \frac{\partial^3}{\partial t^3}G(t,s) \leq 0 \tag{2.9}$$

Proof. We have

$$u''''(t) = 0, \ \ for \ \ t \ \in [0,1],$$
$$u(0) = u'(1) = u''(0) = u'''(1) = 0.$$

Integrating from 0 to 1, we get

$$u(t) = \alpha + \beta t + \gamma t^2 + \delta t^3, \ \ avec \ \ \alpha,\beta,\gamma,\delta \in \mathrm{R}.$$

Then the Green function is the form

$$G(t,s) = \begin{cases} a_1 + a_2 t + a_3 t^2 + a_4 t^3, & 0 \leq t \leq s \leq 1, \\ \\ b_1 + b_2 t + b_3 t^2 + b_4 t^3,, & 0 \leq s \leq t \leq 1, \end{cases}$$

which $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are continuous functions of s.
Thus from the boundary conditions, we obtain

$$G(0.s) = G''(0.s) = 0$$

i.e.

$$a_1 = a_3 = 0$$

and

$$G'(1.s) = G'''(1,s) = 0$$

i.e.

$$b_2 + 2b_3 = b_4 = 0,$$

we pose $c_k = b_k - a_k, (k = 1, 2, 3, 4)$, and the system for linear equations

$$\begin{cases} c_1 + c_2 s + c_3 s^2 + c_4 s^3 = 0 \\ c_2 + 2c_3 s + 3c_4 s^2 = 0 \\ 2c_3 + 6c_4 s = 0 \\ 6c_4 = 1, \end{cases}$$

implies

$$c_1 = -\frac{1}{6}s^3, \quad c_2 = \frac{1}{2}s^2, \quad c_3 = -\frac{1}{2}s, \quad c_4 = \frac{1}{6},$$

and

$$a_2 = s - \frac{1}{2}s^2, \quad a_4 = -\frac{1}{6}, \quad b_1 = -\frac{1}{6}s^3, \quad b_2 = s, \quad b_3 = -\frac{1}{2}s.$$

Then the Green function is given by

$$G(t,s) = \frac{1}{6}\begin{cases} (6t - 3t^2 - s^2)s, & 0 \le s \le t \le 1, \\ (6s - 3s^2 - t^2)t, & 0 \le t \le s \le 1 \end{cases}$$

Then

$$\frac{\partial}{\partial t}G(t,s) = \frac{1}{6}\begin{cases} (6 - 6t)s, & 0 \le s \le t \le 1, \\ (6s - 3s^2 - 3t^2), & 0 \le t \le s \le 1, \end{cases}$$

$$\frac{\partial^2}{\partial t^2}G(t,s) = \frac{1}{6}\begin{cases} -6s, & 0 \le s \le t \le 1, \\ -6t, & 0 \le t \le s \le 1, \end{cases}$$

$$\frac{\partial^3}{\partial t^3}G(t,s) = \begin{cases} 0, & 0 \le s \le t \le 1, \\ -1, & 0 \le t \le s \le 1, \end{cases}$$

The completes proof.

2.2.4 Theorem

Assume that $f \in C([0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0] \times (-\infty,0], [0,+\infty))$ and $f(t,0,0,0,0) \not\equiv 0$, $t \in [0,1]$. Suppose that there exists positive number $d > 0$ such that

$$max\{f(t,u_0,u_1,u_2,u_3) : (t,u_0,u_1,u_2,u_3) \in [0,1] \times [0,d] \times [0,\frac{8}{5}d] \times [-\frac{12}{5}d,0] \times [-\frac{24}{5}d,0]\} \le \frac{24}{5}d. \tag{2.10}$$

Then the problem $(2.1), (2.2)$ has at least one positive solution $u^* \in C^4([0,1])$ such that:

$$0 \le u^*(t) \le d, \quad 0 \le (u^*)'(t) \le \frac{8}{5}d, \quad -\frac{12}{5}d \le (u^*)''(t) \le 0,$$
$$-\frac{24}{5}d \le (u^*)'''(t) \le 0, \quad t \in [0,1].$$

Proof. From (2.8) and after direct computations, we easily get

$$\int_0^1 |G(t,s)|ds = \int_0^1 G(t,s)ds = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{1}{3}t,$$

$$\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right|ds = \int_0^1 \frac{\partial}{\partial t}G(t,s)ds = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3},$$

$$\int_0^1 \left|\frac{\partial^2}{\partial t^2}G(t,s)\right|ds = -\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)ds = -\frac{1}{2}t^2 + t,$$

$$\int_0^1 \left|\frac{\partial^3}{\partial t^3}G(t,s)\right|ds = -\int_0^1 \frac{\partial^3}{\partial t^3}G(t,s)ds = 1 - t.$$

So,

$$\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds = \frac{5}{24}, \quad \max_{0\le t\le 1}\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right|ds = \frac{1}{3},$$

$$\max_{0\le t\le 1}\int_0^1 \left|\frac{\partial^2}{\partial t^2}G(t,s)\right|ds = \frac{1}{2}, \quad \max_{0\le t\le 1}\int_0^1 \left|\frac{\partial^3}{\partial t^3}G(t,s)\right|ds = 1.$$

Now we consider the Banach space $E = C^3([0,1])$ equipped with the norm

$$\|u\| = max\left\{|u|_0, \frac{8}{5}|u'|_0, \frac{12}{5}|u''|_0, \frac{24}{5}|u'''|_0\right\}, \tag{2.11}$$

where $|u|_0 = \max_{0\le t\le 1}|u(t)|$.
For $u \in E$, define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \quad t \in [0,1]. \tag{2.12}$$

Then

$$(Tu)'(t) = \int_0^1 \frac{\partial}{\partial t}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \quad t \in [0,1],$$

$$(Tu)''(t) = \int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \quad t \in [0,1],$$

$$(Tu)'''(t) = \int_t^1 \frac{\partial^3}{\partial t^3}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \quad t \in [0,1].$$

So, $(Tu)(0) = (Tu)'(1) = (Tu)''(0) = (Tu)'''(1) = 0$. Therefore, By Ascoli-Arzela Theorem, it is easy to known that this operator $T : E \longrightarrow E$ is a completely continuous operator. Problem (2.1), (2.2) has a solution $u = u(t)$ if and only if u is a fixed point of operator T defined by (2.12).
Remarque. For prove that the operator T is completely continuous, we apply the same method of the previous proof of Theorem 2.2.2.

Let

$$\Omega_d = \{u \in E : ||u|| < d,\ u(t) \ge 0,\ u'(t) \ge 0,\ u''(t) \le 0,\ u'''(t) \le 0,\ t \in [0,1]\},$$

then Ω_d is a bounded closed convex set of E. If $u \in \partial\Omega_d$ then by (2.11) we have

$$|u_0| \leq d, \quad |u_0'| \leq \frac{8}{5}d, \quad |u_0''| \leq \frac{12}{5}d, \quad |u_0'''| \leq \frac{24}{5}d,$$

which implies

$$0 \leq u(t) \leq d, \quad 0 \leq u'(t) \leq \frac{8}{5}d, \quad -\frac{12}{5}d \leq u''(t) \leq 0, \quad -\frac{24}{5}d \leq u'''(t) \leq 0, \ \ t \in [0,1].$$

Thus (2.10) implies

$$f(t, u(t), u'(t), u''(t), u'''(t)) \leq \frac{24}{5}d, \quad t \in [0,1].$$

Therefore,

$$|Tu|_0 = \max_{0\leq t\leq 1}\left|\int_0^1 G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds\right|$$

$$= \max_{0\leq t\leq 1}\int_0^1 G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds$$

$$|Tu|_0 \leq \frac{24}{5}d\max_{0\leq t\leq 1}\int_0^1 G(t,s)ds = d,$$

$$|(Tu)'|_0 = \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial}{\partial t}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds\right|$$

$$= \max_{0\leq t\leq 1}\int_0^1 \frac{\partial}{\partial t}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds$$

$$\leq \frac{24}{5}d\max_{0\leq t\leq 1}\int_0^1 \frac{\partial}{\partial t}G(t,s)ds = \frac{8}{5}d,$$

$$|(Tu)''|_0 = \max_{0\leq t\leq 1}\left|\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds\right|$$

$$\leq \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^2}{\partial t^2}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))\right|ds$$

$$\leq \frac{24}{5}d\max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^2}{\partial t^2}G(t,s)\right|ds = \frac{12}{5}d,$$

$$|(Tu)'''|_0 = \max_{0\leq t\leq 1}\left|\int_t^1 \frac{\partial^3}{\partial t^3}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds\right|$$

$$\leq \max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^3}{\partial t^3}G(t,s)f(s,u(s),u'(s),u''(s),u'''(s))\right|ds$$

$$\leq \frac{24}{5}d\max_{0\leq t\leq 1}\int_0^1 \left|\frac{\partial^3}{\partial t^3}G(t,s)\right|ds = \frac{24}{5}d.$$

Thus

$$||Tu|| = max\left\{|Tu|_0, \frac{8}{5}|(Tu)'|_0, \frac{12}{5}|(Tu)''|_0, \frac{24}{5}|(Tu)'''|_0\right\} \leq d,$$

i.e., $Tu \in \partial\Omega_d$. So, by Leray-Schauder fixed point theorem, T has a fixed point $u^* \in \Omega_d$, which is a solution of the problem $(2.1),(2.2)$. Noticing that $f(t,0,0,0,0) \not\equiv 0$. So $u = 0$ is not a solution of the problem $(2.1),(2.2)$, therefore $|u^*|_0 > 0$. From (2.9) we know that $u^*(t)$ is nondecreasing and concave on $[0,1]$, thus $u^*(t) \geq t|u^*|_0 > 0$ for $t \in [0,1]$. So, $u^*(t)$ is a positive solution of the problem $(2.1),(2.2)$. This completes the proof.

2.3 Application

Consider the following problem FBVP

$$\begin{aligned} &u^{(4)}(t) = \tfrac{\sqrt{t}}{26}u(t) + \tfrac{t^{15}}{2}u^{'}(t) - \tfrac{t^{11}}{4}u^{''}(t) - \tfrac{\sqrt[3]{t}}{59}u^{'''}(t) + t^3 + 1, \\ &u(0) = u^{'}(1) = u^{''}(0) = u^{'''}(1) = 0. \end{aligned} \tag{2.13}$$

Set

$$f(t,u_0,u_1,u_2,u_3) = \frac{\sqrt{t}}{26}u_0(t) + \frac{t^{15}}{2}u_1(t) - \frac{t^{11}}{4}u_2(t) - \frac{\sqrt[3]{t}}{59}u_3(t) + t^3 + 1,$$

and

$$a_0(t) = \frac{\sqrt{t}}{26},\ a_1(t) = t^{15},\ a_2(t) = \frac{t^{11}}{4},\ a_3(t) = \frac{\sqrt[3]{t}}{59},\ a_4(t) = t^3 + 2,$$

It is easy to prove that $a_i \in L^1[0,1],\ i = 0,1,2,3,4,$ are nonnegative functions, $f(t,0,0,0,0) = t^3 + 1 \neq 0$.

Moreover, we have

$$\begin{aligned} B &= \frac{1}{3}\int_0^1 a_0(s)ds + \frac{1}{2}\int_0^1 a_1(s)ds + \int_0^1 a_2(s)ds + \int_0^1 a_3(s)ds \\ &= \frac{1}{3}\int_0^1 \frac{\sqrt{s}}{26}ds + \frac{1}{2}\int_0^1 s^{15}ds + \int_0^1 \frac{s^{11}}{4}ds + \int_0^1 \frac{\sqrt[3]{s}}{59}ds \\ &= \frac{1}{117} + \frac{1}{32} + \frac{1}{48} + \frac{3}{236} \simeq 0,0732 < 1, \end{aligned}$$

and for any

$$(t,u_0,u_1,u_2,u_3) \in [0,1] \times [0,\frac{\rho}{3}] \times [0,\frac{\rho}{2}] \times [-\rho,0] \times [0,\rho],$$

and f satisfies

$$f(t,u_0,u_1,u_2,u_3) \leq a_0(t)u_0 + a_1(t)u_1 - a_2(t)u_2 - a_3(t)u_3 + a_4(t).$$

where

$$A = \int_0^1 a_4(s)ds = \frac{9}{4},\ \rho = A(1-B)^{-1} \simeq 2,4277.$$

Hence, by Theorem 2.2.2, the FBVP (2.13) has at least one positive solution u^* in $C^4([0,1])$ such that

$$3\max_{0\leq t\leq 1} u^*(t) \leq 2\max_{0\leq t\leq 1}(u^*)'(t) \leq \max_{0\leq t\leq 1}[-(u^*)''(t)] \leq \max_{0\leq t\leq 1}[-(u^*)'''(t)] \leq \rho.$$

Chapter 3

Solvability For A Nonlinear Fourth-Order Three-Point Boundary Value Problem

3.1 Introduction

In this chapter, we study the existence of nontrivial solution for the fourth-order three-point boundary value problem having the following form

$$u^{(4)}(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = 0, \quad u^{''}(0) = u^{'''}(0) = 0, \quad u^{'}(1) = \alpha u^{'}(\eta), \tag{3.2}$$

where $\eta \in (0, 1)$, $\alpha \in \mathbf{R}$, $\alpha \neq 1$, $f \in C([0, 1] \times \mathbf{R}, \mathbf{R})$. By using Leray-Schauder nonlinear alternative, we prove the existence of at least one solution of the posed problem. As an application, we also given some examples to illustrate the obtained results.

3.2 Preliminaries

Let $E = C([0, 1])$ with the norm given by $\|y\| = \sup_{t\in[0,1]} |y(t)|$, for any $y \in E$. A solution $u(t)$ of the BVP $(3.1) - (3.2)$ is called nontrivial solution if $u(t) \neq 0$. To get our results, we need the following lemma.

3.2.1 Lemma

Let $y \in C([0, 1])$, $\alpha \neq 1$, then boundary value problem

$$u^{(4)}(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \quad u^{''}(0) = u^{'''}(0) = 0, \quad u^{'}(1) = \alpha u^{'}(\eta),$$

has a unique solution

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 y(s)\mathrm{d}s.$$

Proof. Rewriting the differential equation as $u^{(4)}(t) = -y(t)$ and integrating four times from 0 to t, we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t^3}{6}c + \frac{t^2}{2}c_1 + tc_2 + c_3. \tag{3.3}$$

By the boundary conditions (3.2), we have

$$u(0) = 0, \quad u''(0) = u'''(0) = 0, \ \ i.e.\ \ c_1 = c_3 = c = 0,$$

and $u'(1) = \alpha u'(\eta)$, and thus we get

$$c_2 = \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 y(s)\mathrm{d}s. \tag{3.4}$$

Using the equations (3.4) and (3.3), we obtain

$$u(t) = -\frac{1}{6}\int_0^t (t-s)^3 y(s)\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 y(s)\mathrm{d}s - \frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 y(s)\mathrm{d}s.$$

Define the integral operator $T : E \longrightarrow E$, by

$$Tu(t) = -\frac{1}{6}\int_0^t (t-s)^3 f(s,u(s))\mathrm{d}s + \frac{t}{2(1-\alpha)}\int_0^1 (1-s)^2 f(s,u(s))\mathrm{d}s -$$

$$\frac{\alpha t}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 f(s,u(s))\mathrm{d}s. \tag{3.5}$$

By Lemma $3.2.1$, the BVP $(3.1) - (3.2)$ has a solution if and only if the operator T has a fixed point in E. So we only need to find a fixed point of T in E. By Ascoli-Arzela theorem, we can prove that T is a completely continuous operator.
Remarque. For prove that the operator T is completely continuous, we apply the same method of the previous proof of chapter 2.

3.3 Existence of Nontrivial Solutions

In this section, we prove the existence of a nontrivial solution for the BVP $(3.1) - (3.2)$. Suppose that $f \in C([0,1] \times \mathbf{R}, \mathbf{R})$.

3.3.1 Theorem

Suppose that $f(t,0) \neq 0$, $\alpha \neq 1$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ \ (t,x) \in [0,1] \times \mathbf{R},$$

$$\frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s < 1.$$

Then the problem $(3.1) - (3.2)$ has at least one nontrivial solution $u^* \in C([0,1])$.

Proof. Let

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s,$$

$$N = \frac{1}{6}\int_0^1 (1-s)^3 h(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 h(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 h(s)\mathrm{d}s.$$

Then $M < 1$. Since $f(t,0) \neq 0$, there exists an interval $[a,b] \subset [0,1]$ such that $\min_{a\leq t\leq b}|f(t,0)| > 0$. As $h(t) \geq |f(t,0)|$, a.e. $t \in [0,1]$, we know that $N > 0$.

Let $A = N(1-M)^{-1}$ and $\Omega = \{u \in E : \|u\| < A\}$. Let $u \in \partial\Omega$ and $\lambda > 1$ be such that $Tu = \lambda u$. Then

$$\begin{aligned}
\lambda A &= \lambda\|u\| = \|Tu\| = \max_{0\leq t\leq 1}|(Tu)(t)| \\
&\leq \frac{1}{6}\max_{0\leq t\leq 1}\int_0^t (t-s)^3|f(s,u(s))|\mathrm{d}s + \max_{0\leq t\leq 1}\left|\frac{t}{2(1-\alpha)}\right|\int_0^1 (1-s)^2|f(s,u(s))|\mathrm{d}s \\
&+ \max_{0\leq t\leq 1}\left|\frac{\alpha t}{2(1-\alpha)}\right|\int_0^\eta (\eta-s)^2|f(s,u(s))|\mathrm{d}s \\
&\leq \frac{1}{6}\int_0^1 (1-s)^3|f(s,u(s))|\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2|f(s,u(s))|\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|} \\
&\times \int_0^\eta (\eta-s)^2|f(s,u(s))|\mathrm{d}s \\
&\leq [\frac{1}{6}\int_0^1 (1-s)^3 k(s)|u(s)|\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)|u(s)|\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|} \\
&\times \int_0^\eta (\eta-s)^2 k(s)|u(s)|\mathrm{d}s] + [\frac{1}{6}\int_0^1 (1-s)^3 h(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 h(s)\mathrm{d}s \\
&+ \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 h(s)\mathrm{d}s] \\
&= M\|u\| + N.
\end{aligned}$$

Therefore,

$$\lambda \leq M + \frac{N}{A} = M + \frac{N}{N(1-M)^{-1}} = M + (1-M) = 1.$$

This contradicts $\lambda > 1$. By Leray-Schauder Nonlinear Alternative, T has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0) \neq 0$, the problem $(3.1) - (3.2)$ has a nontrivial solution $u^* \in E$.

This completes the proof.

3.3.2 Theorem

Suppose that $f(t,0) \neq 0$, $\alpha < 1$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbf{R}.$$

If one of the following conditions is fulfilled

(1) There exists a constant $p>1$ such that

$$\int_0^1 k^p(s)\mathrm{d}s < [\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}]^p,\ (\frac{1}{p}+\frac{1}{q}=1).$$

(2) There exists a constant $\mu>-1$ such that

$$k(s)\leq \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}s^{\mu},\quad a.e.\quad s\in[0,1],$$

$$meas\{s\in[0,1]:k(s)<\frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}s^{\mu}\}>0.$$

(3) There exists a constant $\mu>-3$ such that

$$k(s)\leq \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^{\mu},\quad a.e.\quad s\in[0,1],$$

$$meas\{s\in[0,1]:k(s)<\frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^{\mu}\}>0.$$

(4) k satisfies

$$k(s)\leq \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)},\quad a.e.\quad s\in[0,1],$$

$$meas\{s\in[0,1]:k(s)<\frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}\}>0.$$

Then the problem $(3.1)-(3.2)$ has at least one nontrivial solution $u^*\in E$.

Proof. Let M be defined as in the proof of Theorem 2.3.1. To prove Theorem 2.3.2, we only need to prove that $M<1$. Since $\alpha<1$, we have

$$M=\frac{1}{6}\int_0^1(1-s)^3k(s)\mathrm{d}s+\frac{1}{2(1-\alpha)}\int_0^1(1-s)^2k(s)\mathrm{d}s+\frac{|\alpha|}{2(1-\alpha)}\int_0^{\eta}(\eta-s)^2k(s)\mathrm{d}s.$$

(1) Using the Hölder inequality, we have

$$M\leq[\int_0^1 k^p(s)\mathrm{d}s]^{1/p}\{\frac{1}{6}[\int_0^1(1-s)^{3q}\mathrm{d}s]^{1/q}+\frac{1}{2(1-\alpha)}[\int_0^1(1-s)^{2q}\mathrm{d}s]^{1/q}+\frac{|\alpha|}{2(1-\alpha)}\times$$

$$[\int_0^{\eta}(\eta-s)^{2q}\mathrm{d}s]^{1/q}\}$$

$$\leq[\int_0^1 k^p(s)\mathrm{d}s]^{1/p}[\frac{1}{6}(\frac{1}{1+3q})^{1/q}+\frac{1}{2(1-\alpha)}(\frac{1}{1+2q})^{1/q}+\frac{|\alpha|}{2(1-\alpha)}(\frac{\eta^{1+2q}}{1+2q})^{1/q}]$$

$$<\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}\times$$

$$\frac{(1-\alpha)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}=1.$$

(2) In this case, we have

$$M < \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}[\frac{1}{6}\int_0^1 (1-s)^3 s^\mu \mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 s^\mu \mathrm{d}s +$$

$$\frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2 s^\mu \mathrm{d}s]$$

$$\leq \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}[\frac{1}{(1+\mu)(2+\mu)(3+\mu)(4+\mu)} +$$

$$\frac{1}{(1-\alpha)}\frac{1}{(1+\mu)(2+\mu)(3+\mu)} + \frac{|\alpha|}{(1-\alpha)}\frac{\eta^{3+\mu}}{(1+\mu)(2+\mu)(3+\mu)}]$$

$$= \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}.\frac{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)} = 1.$$

(3) In this case, we have

$$M < \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s +$$

$$\frac{|\alpha|}{2(1-\alpha)}\int_0^\eta (\eta-s)^2(1-s)^\mu \mathrm{d}s]$$

$$\leq \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}[\frac{1}{6}\int_0^1 (1-s)^{3+\mu}\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s +$$

$$\frac{|\alpha|}{2(1-\alpha)}\int_0^1 (1-s)^{2+\mu}\mathrm{d}s]$$

$$= \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}.\frac{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}{6(1-\alpha)(3+\mu)(4+\mu)} = 1.$$

(4) In this case, we have

$$M < \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}[\frac{1}{6}\int_0^1 (1-s)^3\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2\mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\times$$

$$\int_0^\eta (\eta-s)^2\mathrm{d}s]$$

$$= \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|\eta^3)}.\frac{(1-\alpha)+4(1+|\alpha|\eta^3)}{24(1-\alpha)} = 1.$$

This completes the proof.

3.3.3 Theorem

Suppose that $f(t,0) \neq 0$, $\alpha > 1$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1] \times \mathbf{R}.$$

If one of the following conditions holds

(1) There exists a constant $p > 1$ such that

$$\int_0^1 k^p(s)\mathrm{d}s < [\frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})}]^p, \quad (\frac{1}{p}+\frac{1}{q}=1).$$

(2) There exists a constant $\mu > -1$ such that

$$k(s) \le \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}s^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}s^{\mu}\} > 0.$$

(3) There exists a constant $\mu > -3$ such that

$$k(s) \le \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}(1-s)^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}(1-s)^{\mu}\} > 0.$$

(4) k satisfies

$$k(s) \le \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}\} > 0.$$

Then the problem $(3.1)-(3.2)$ has at least one nontrivial solution $u^* \in E$.

Proof. Let M be defined as in the proof of Theorem 3.3.1. To prove Theorem 3.3.3, we only need to prove that $M < 1$. Since $\alpha > 1$, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^{\eta} (\eta-s)^2 k(s)\mathrm{d}s.$$

(1) Using the Hölder inequality, we have

$$\begin{aligned} M \le\ & [\int_0^1 k^p(s)\mathrm{d}s]^{1/p}\{\frac{1}{6}[\int_0^1 (1-s)^{3q}\mathrm{d}s]^{1/q} + \frac{1}{2(\alpha-1)}[\int_0^1 (1-s)^{2q}\mathrm{d}s]^{1/q} + \frac{\alpha}{2(\alpha-1)}\times \\ & [\int_0^{\eta} (\eta-s)^{2q}\mathrm{d}s]^{1/q}\} \\ \le\ & [\int_0^1 k^p(s)\mathrm{d}s]^{1/p}[\frac{1}{6}(\frac{1}{1+3q})^{1/q} + \frac{1}{2(\alpha-1)}(\frac{1}{1+2q})^{1/q} + \frac{\alpha}{2(\alpha-1)}(\frac{\eta^{1+2q}}{1+2q})^{1/q}] \\ <\ & \frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})}\times \\ & \frac{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha\eta^{(1+2q)/q})}{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}} = 1. \end{aligned}$$

(2) In this case, we have

$$M < \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}[\frac{1}{6}\int_0^1(1-s)^3s^\mu \mathrm{d}s+\frac{1}{2(\alpha-1)}\int_0^1(1-s)^2s^\mu \mathrm{d}s+$$

$$\frac{\alpha}{2(\alpha-1)}\int_0^\eta(\eta-s)^2s^\mu \mathrm{d}s]$$

$$= \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}.\frac{(\alpha-1)+(4+\mu)(1+\alpha\eta^{3+\mu})}{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)} = 1.$$

(3) In this case, we have

$$M < \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}[\frac{1}{6}\int_0^1(1-s)^{3+\mu}\mathrm{d}s+\frac{1}{2(\alpha-1)}\int_0^1(1-s)^{2+\mu}\mathrm{d}s+$$

$$\frac{\alpha}{2(\alpha-1)}\int_0^\eta(\eta-s)^2(1-s)^\mu \mathrm{d}s]$$

$$\leq \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}[\frac{1}{6}\int_0^1(1-s)^{3+\mu}\mathrm{d}s+\frac{1}{2(\alpha-1)}\int_0^1(1-s)^{2+\mu}\mathrm{d}s+$$

$$\frac{\alpha}{2(\alpha-1)}\int_0^1(1-s)^{2+\mu}\mathrm{d}s]$$

$$= \frac{6(\alpha-1)(3+\mu)(4+\mu)}{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}.\frac{(\alpha-1)(3+\mu)+3(1+\alpha)(4+\mu)}{6(\alpha-1)(3+\mu)(4+\mu)} = 1.$$

(4) In this case, we have

$$M < \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}[\frac{1}{6}\int_0^1(1-s)^3\mathrm{d}s+\frac{1}{2(\alpha-1)}\int_0^1(1-s)^2\mathrm{d}s+\frac{\alpha}{2(\alpha-1)}\times$$

$$\int_0^\eta(\eta-s)^2\mathrm{d}s]$$

$$= \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha\eta^3)}.\frac{(\alpha-1)+4(1+\alpha\eta^3)}{24(\alpha-1)} = 1.$$

This completes the proof.

3.3.4 Corollary

Suppose $f(t,0) \neq 0$, $\alpha < 1$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbf{R}.$$

If one of following conditions is holds

(1) There exists a constant $p > 1$ such that

$$\int_0^1 k^p(s)ds < [\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+|\alpha|)}]^p, \ (\frac{1}{p}+\frac{1}{q}=1).$$

(2) There exists a constant $\mu > -1$ such that

$$k(s) \leq \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|)} s^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|)} s^{\mu}\} > 0.$$

(3) k satisfies

$$k(s) \leq \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24(1-\alpha)}{(1-\alpha)+4(1+|\alpha|)}\} > 0.$$

Then the problem $(3.1)-(3.2)$ has at least one nontrivial solution $u^* \in E$.

Proof. In this case, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\int_0^{\eta} (\eta-s)^2 k(s)\mathrm{d}s$$

$$\leq \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s$$

$$= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1+|\alpha|}{2(1-\alpha)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s.$$

Now, the proof follows, by using the same method as the one used in the proof of Theorem 2.3.2. The proof is complete.

3.3.5 Corollary

Suppose that $f(t,0) \neq 0$, $\alpha > 1$, and there exist nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbf{R}.$$

If one of the following conditions holds

(1) There exists a constant $p > 1$ such that

$$\int_0^1 k^p(s)ds < [\frac{6(\alpha-1)(1+2q)^{1/q}(1+3q)^{1/q}}{(\alpha-1)(1+2q)^{1/q}+3(1+3q)^{1/q}(1+\alpha)}]^p, \quad (\frac{1}{p}+\frac{1}{q}=1).$$

(2) There exists a constant $\mu > -1$ such that

$$k(s) \leq \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha)} s^{\mu}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{(\alpha-1)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(\alpha-1)+(4+\mu)(1+\alpha)} s^{\mu}\} > 0.$$

(3) k satisfies

$$k(s) \leq \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha)}, \quad a.e. \quad s \in [0,1],$$

$$meas\{s \in [0,1] : k(s) < \frac{24(\alpha-1)}{(\alpha-1)+4(1+\alpha)}\} > 0.$$

Then the problem $(3.1)-(3.2)$ has at least one nontrivial solution $u^* \in E$.

Proof. In this case, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s$$

$$\leq \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{\alpha}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s$$

$$= \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1+\alpha}{2(\alpha-1)}\int_0^1 (1-s)^2 k(s)\mathrm{d}s.$$

The rest of the proof follows in the same way as in the proof of Theorem 3.3.3. This completes the proof.

3.4 Examples

In order to illustrate the above results, we consider some examples.

3.4.1 Example

Consider the three-point boundary value problem

$$\begin{aligned} &u^{(4)} + \tfrac{t}{\sqrt{2}} u \sin u - t - 2 = 0, \quad 0 < t < 1, \\ &u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -3u'(1/2). \end{aligned} \tag{3.6}$$

Set $\eta = 1/2$, $\alpha = -3 \neq 1$, and

$$f(t,x) = \frac{t}{\sqrt{2}} x \sin x - t - 2,$$

$$k(t) = t, \quad h(t) = t + 2,$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \leq k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbf{R},$$

Moreover, we have

$$M = \frac{1}{6}\int_0^1 (1-s)^3 k(s)\mathrm{d}s + \frac{1}{2|1-\alpha|}\int_0^1 (1-s)^2 k(s)\mathrm{d}s + \frac{|\alpha|}{2|1-\alpha|}\int_0^\eta (\eta-s)^2 k(s)\mathrm{d}s$$

$$= \frac{1}{6}\int_0^1 (1-s)^3 s\mathrm{d}s + \frac{1}{8}\int_0^1 (1-s)^2 s\mathrm{d}s + \frac{3}{8}\int_0^{1/2} (\frac{1}{2}-s)^2 s\mathrm{d}s = 0.019 < 1.$$

Hence, by Theorem (3.3.1), the BVP (3.6) has at least one nontrivial solution u^* in E.

3.4.2 Example

Consider the three-point boundary value problem

$$\begin{gathered} u^{(4)} + \tfrac{3}{25}(7+t)u - e^t + 1 = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -2u'(1/4). \end{gathered} \tag{3.7}$$

Set $\eta = 1/4$, $\alpha = -2 < 1$, and

$$f(t,x) = \frac{3}{25}(7+t)x - e^t + 1,$$

$$k(t) = \frac{1}{2}(7+t), \quad h(t) = e^t + 1.$$

It is easy to prove that $k, h \in L^1[0,1]$ are nonnegative functions and

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e. \ (t,x) \in [0,1] \times \mathbf{R}.$$

Let $p = q = 2$, such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 k^p(s)\mathrm{d}s = \int_0^1 \frac{1}{4}(7+s)^2 \mathrm{d}s = \frac{169}{12}.$$

Moreover, we have

$$[\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}]^p = 49.454.$$

Therefore,

$$\int_0^1 k^p(s)\mathrm{d}s < [\frac{6(1-\alpha)(1+2q)^{1/q}(1+3q)^{1/q}}{(1-\alpha)(1+2q)^{1/q} + 3(1+3q)^{1/q}(1+|\alpha|\eta^{(1+2q)/q})}]^p.$$

Hence, by Theorem 3.3.2 (1), the BVP (3.7) has at least one nontrivial solution u^* in E.

3.4.3 Example

Consider the three-point boundary value problem

$$\begin{gathered} u^{(4)} + \tfrac{tu^2}{9(5+u)} \cos u - e^t - 1 = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -4u'(1/3). \end{gathered} \tag{3.8}$$

Set $\eta = 1/3$, $\alpha = -4 < 1$, and

$$f(t,x) = \frac{tx^2}{9(5+x)} \cos x - e^t - 1,$$

$$k(t) = \frac{1}{9}t, \quad h(t) = e^t + 1.$$

It is easy to prove that $k, h \in L^1[0, 1]$ are nonnegative functions and

$$|f(t, x)| \leq k(t)|x| + h(t), \quad a.e. \ (t, x) \in [0, 1] \times \mathbf{R}.$$

Let $\mu = 1 > -1$, then

$$\frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})} = 58.559.$$

Therefore,

$$k(s) = \frac{1}{9}s < 58.559.s,$$

$$meas\{s \in [0, 1] : k(s) < \frac{(1-\alpha)(1+\mu)(2+\mu)(3+\mu)(4+\mu)}{(1-\alpha)+(4+\mu)(1+|\alpha|\eta^{3+\mu})}s^{\mu}\} > 0.$$

Hence, by Theorem 3.3.2 (2), the BVP (3.8) has at least one nontrivial solution u^* in E.

3.4.4 Example

Consider the three-point boundary value problem

$$\begin{aligned} &u^{(4)} + \tfrac{5u^3}{8(1+u)(1-t)^{-2}} \sin u + t^4 - 3 = 0, \quad 0 < t < 1, \\ &u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -6u'(1/2). \end{aligned} \tag{3.9}$$

Set $\eta = 1/2$, $\alpha = -6 < 1$, and

$$f(t, x) = \frac{5x^3}{8(1+x)(1-t)^{-2}} \sin x + t^4 - 3,$$

$$k(t) = \frac{5}{8(1-t)^{-2}}, \quad h(t) = t^4 + 3.$$

It is easy to prove that $k, h \in L^1[0, 1]$ are nonnegative functions and

$$|f(t, x)| \leq k(t)|x| + h(t), \quad a.e. \ (t, x) \in [0, 1] \times \mathbf{R}.$$

Let $\mu = 2 > -3$, then

$$\frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)} = \frac{1260}{161}.$$

Therefore,

$$k(s) = \frac{5}{8}(1-s)^2 < \frac{1260}{161}(1-s)^2,$$

$$meas\{s \in [0, 1] : k(s) < \frac{6(1-\alpha)(3+\mu)(4+\mu)}{(1-\alpha)(3+\mu)+3(1+|\alpha|)(4+\mu)}(1-s)^{\mu}\} > 0.$$

Hence, by Theorem 3.3.2 (3), the BVP (3.9) has at least one nontrivial solution u^* in E.

3.4.5 Example

Consider the three-point boundary value problem

$$\begin{gathered} u^{(4)} + \tfrac{tu^2}{5(3+u)} + e^t - 3 = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u''(0) = u'''(0) = 0, \quad u'(1) = -5u'(1/5). \end{gathered} \tag{3.10}$$

Set $\eta = 1/5$, $\alpha = -5 < 1$, and

$$f(t, x) = \frac{tx^2}{5(3+x)} + e^t - 3,$$

$$k(t) = \frac{t}{5}, \quad h(t) = e^t + 3.$$

It is easy to prove that $k, h \in L^1[0, 1]$ are nonnegative functions and

$$|f(t, x)| \leq k(t)|x| + h(t), \quad a.e. \ (t, x) \in [0, 1] \times \mathbf{R}.$$

Moreover, we have

$$\frac{24(1-\alpha)}{(1-\alpha) + 4(1+|\alpha|\eta^3)} = \frac{1800}{127}.$$

Therefore,

$$k(s) = \frac{s}{5} < \frac{1800}{127}, \quad s \in [0, 1],$$

$$meas\{s \in [0, 1] : k(s) < \frac{24(1-\alpha)}{(1-\alpha) + 4(1+|\alpha|\eta^3)}\} > 0.$$

Hence, by Theorem 3.3.2 (4), the BVP (3.10) has at least one nontrivial solution u^* in E.

Remark

We can give similar examples for Theorem 3.3.3, Corollary 3.3.4 and Corollary 3.3.5.

Conclusions

In this thesis, we have proved the existence of at least one nontrivial solution and the existence of positive solution for nonlinear fourth-order boundary value problem.

In this work, we concluded that this method was used in the chapter 3, we can apply to any boundary value problem with posed all sufficient conditions to prove the existence of at least one nontrivial solution. For example, we can applied to boundary value problem involving integral boundary conditions.

We concluded that we could apply the method the Banach contraction principle for prove existence and uniqueness of solution of fourth order boundary value problem in the problem is presented in sam chapter.

Also, we concluded that the method was used in chapter 2, we can apply to nonlinear sixth-order boundary value problem, with all conditions sufficient for Green function.

We have also come to the conclusion that putting a problem in form $-u^{(4)}(t) = q(t)f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1$, lead to prove existence of negative solution for fourth order boundary value problem in same chapter. That is, by an inverse problem.

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